Contents

1 Complex Analysis

A function defined on a set of complex numbers S is a rule which assigns to each z in S a unique complex number w, denoted:

$$w = f(z)$$

Here z takes on values in S, and is called a complex variable. The set S is the domain of definition of $f(z)$.

Let u and v be the real and imaginary parts of w. Since w depends on $z = x + iy$, and therefore on x and y, so do u and v. We can therefore write

$$w = f(z) = u(x, y) + iv(x, y)$$

A complex function $f(z)$ is equivalent to two real functions $u(x, y)$ and $v(x, y)$, each depending on x and y.

Note that in order to have a well defined complex function, we must obey the **unique** mapping fron z to $f(z)$. For example, $\bar{z}$ (the complex conjugate), is a valid function, since the mapping $x + iy \to x - iy$ produces a unique mapping for each (x, y).

Consider the function $f(z) = z^2 + 3z$. What if we want to determine $u(x, y)$ and $v(x, y)$?

We can write this out:

$$\begin{aligned} f(z) &= (x + iy)^2 + 3(x + iy) \\ &= \left(x^2 + 2ixy - y^2\right) + 3(x + iy) \\ &= \left(x^2 + 3x - y^2\right) + i(2xy + 3y) \end{aligned}$$

And thus we have that $u(x, y) = x^2 + 3x - y^2$ and $v(x, y) = 2xy + 3y$.

1.1 Analytic Functions

A function $f(z)$ is **analytic** in a region of the complex plane if it has a *unique* derivative at every point in the region.

What does it mean for $f(z)$ to have a unique derivative? Let us compare this to a function $g(x)$ of a real variable. Let's say that the function is smooth, but it has a sharp point at x_0. At every other point, $g(x)$ has a derivative, but at x_0, $g(x)$ does not have a derivative, the left and right handed limits at x_0 do not match. Let us now generalize this idea to functions in the complex plane.

This case is more complicated because instead of having just two directions of approach, we have a 2D plane, so we can approach from an infinite number of directions. In this case, we need every one of these infinitely many directional limits to match, this is how we define a unique derivative for a complex function.

How do we define the derivative of a complex function $f(z)$? We define it in the same way as we do for real functions:

$$f'(z) = \frac{df}{dz} = \lim_{\Delta z \to 0} \frac{\Delta f}{\Delta z}$$

where $\Delta f = f(z - \Delta z)$ and $\Delta z = \Delta x + i\Delta y$.

The requirement for analyticity is incredibly stict, and is much more difficult to satisfy than the real counterpart.

Let's do a couple of examples. Consider the function $f(z) = z^2$. Is this analytic?

Doing this from first principles:

$$\begin{aligned}
\frac{d}{dz}\left(z^2\right) &= \lim_{\Delta z \to 0} \frac{(z+\Delta z)^2 - z^2}{\Delta z} \\
&= \lim_{\Delta z \to 0} \frac{z^2 + 2z\Delta z + (\Delta z)^2 - z^2}{\Delta z} \\
&= \lim_{\Delta z \to 0} \frac{2z\Delta z + (\Delta z)^2}{\Delta z} \\
&= \lim_{\Delta z \to 0} 2z + \Delta z \\
&= 2z
\end{aligned}$$

Thus we see that, since the result is independent of the direction of Δz, $f(z) = z^2$ is indeed analytic.

What about the function $f(z) = \bar{z}$?

$$\begin{aligned}
\frac{df}{dz} &= \lim_{\Delta z \to 0} \frac{\overline{(z+\Delta z)} - \bar{z}}{\Delta z} \\
&= \lim_{\Delta x, \Delta y \to 0} \frac{\overline{(x+\Delta x) + i(y+\Delta y)} - \overline{(x+iy)}}{\Delta x + i\Delta y} \\
&= \lim_{\Delta x, \Delta y \to 0} \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y}
\end{aligned}$$

We can vary both Δx and Δy independently, and we note that the case where we approach along $\Delta y = 0$ (the x-axis) and the case where we approach along $\Delta x = 0$ (the y-axis) provide different derivatives ($+1$ and -1 respectively). Therefore, the complex conjugate is not analytic.

What is the general condition a function $f(z)$ has to satisfy in order to be analytic?

Theorem 1.1. *Cauchy-Riemann conditions. If $f(z) = u(x,y) + iv(x,y)$ is analytic in a given region, then in that region:*

$$\begin{aligned}
\frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\
\frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x}
\end{aligned}$$

These are the Cauchy-Riemann conditions.

Proof. We want to show that $\frac{df}{dz}$ is defined and unique for all x and y.

Starting from the definition:

$$\frac{df}{dz} = \lim_{\delta z \to 0} \frac{\delta f}{\delta z}$$

Now we note that $\delta z = \delta x + i\delta y$, and therefore $\delta f = \delta u + i\delta v$. Thus:

$$\frac{\delta f}{\delta z} = \frac{\delta u + i\delta v}{\delta x + i\delta y}$$

We now consider two approaches to the point (x, y), one that is parallel to the x-axis, and the other that is parallel to the y-axis. We will show that the requirement that these two approaches are equivalent will imply the Cauchy-Riemann conditions.

Approaching along $\delta y = 0$, We have that

$$\begin{aligned}\lim_{\delta z \to 0} \frac{\delta f}{\delta z} &= \lim_{\delta x \to 0} \frac{\delta u + i\delta v}{\delta x} \\ &= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\end{aligned}$$

Approaching along $\delta x = 0$:

$$\begin{aligned}\lim_{\delta z \to} \frac{\delta f}{\delta z} &= \lim_{\delta y \to 0} \frac{\delta u + i\delta v}{i\delta y} \\ &= -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}\end{aligned}$$

Now we enforce the condition that these two are equal:

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

Now separating the real and imaginary components:

$$\begin{aligned}\frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x}\end{aligned}$$

Which are the Cauchy-Riemann conditions. □

Now we note that the other direction is true, Cauchy-Riemann being satisfied imply analyticity.

Theorem 1.2. *If $u(x, y)$ and $v(x, y)$ and their partial derivatives with respect to x and y are continuous and satisfy the Cauchy-Riemann conditions in a region, then $f(z)$ is analytic at all points inside the region (though not necessarily on the border).*

Proof. By definition:

$$\begin{aligned}\delta f &= \frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial y}\delta y \\ &= \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\delta x + \left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\delta y\end{aligned}$$

From this:

$$\frac{\delta f}{\delta z} = \frac{\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\delta x + \left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\delta y}{\delta x + i\delta y}$$

Now let us impose the Cauchy-Riemann conditions. We can replace terms in the numerator:

$$\begin{aligned}\frac{\delta f}{\delta z} &= \frac{\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\delta x + \left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\delta y}{\delta x + i\delta y} \\ &= \frac{\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\delta x + \left(-\frac{\partial v}{\partial x} + i\frac{\partial u}{\partial x}\right)\delta y}{\delta x + i\delta y}\end{aligned}$$

We can now see that we are left with

$$\begin{aligned}\frac{\delta f}{\delta z} &= \frac{\left(\frac{\partial u}{\partial x}\right)(\delta x + i\delta y) + i\frac{\partial v}{\partial x}(\delta x + i\delta y)}{\delta x + i\delta y} \\ &= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\end{aligned}$$

Note that this is independent of the direction chosen, and therefore we have analyticity. □

Let us introduce some more definitions. A **regular point** of $f(z)$ is a point at which $f(z)$ is analytic. A **singular point** of $f(z)$ is a point at which $f(z)$ is not analytic. It is denoted an **isolated singular point** if $f(z)$ is analytic everywhere else inside some small circle about the singular point.

Theorem 1.3. *If $f(z)$ is analytic in a region, then it has derivatives of all orders at points inside the region, which are then also analytic functions in that region.*

We state this theorem without proof. We can use this theorem to infer an important conclusion about the functions $u(x, y)$ and $v(x, y)$, they must have partial derivatives to all orders.

Consider the Cauchy-Riemann conditions:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \tag{1}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \tag{2}$$

Suppose we take the partial with respect to x of Equation 1, and the partial with respect to y of Equation 2, and add them:

$$\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial x^2} = 0$$

This is Laplace's equation in two dimensions. Any function u that is part of a complex function $f(z)$ must be a solution to the 2D Laplace's equation. Similarly, we can take $\frac{\partial}{\partial y}$ of Equation 1 and subtract the $\frac{\partial}{\partial x}$ of Equation 2, and we have the condition:

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

We need both of these to be true simultaneously, and is in part the reason that analyticity is a very strict condition. Note that solutions to Laplace's equation are known as *harmonic* functions, and thus $u(x, y)$ and $v(x, y)$ must be **conjugate harmonic functions**.

Let us now study some real valued functions generalized to the complex plane. Some of these will be analytic on the entire complex plane, denoting them *entire* functions. For example, all polynomials of the form

$$f(z) = c_0 + c_1 z + c_2 z^2 + \ldots c_n z^n$$

are entire functions.

A quotient of two polynomials $p(z)$ and $q(z)$ is called a rational function:

$$f(z) = \frac{p(z)}{q(z)}$$

These are analytic for every z for which $q(z)$ is nonzero. A rational function of the form

$$\frac{c}{(z - z_0)^m}$$

where c is a constant, is known as a *partial fraction.*

Now let us consider the exponential function:

$$\begin{aligned} e^z &= e^{x+iy} \\ &e^x e^{iy} \\ &= e^x (\cos y + i \sin y) \end{aligned}$$

The exponential function is an entire function.

The trig functions are defined the same as they are on the real line, except that x is replaced with $x + iy$. The sine and cosine functions are entire, tangent and secant are analytic except where cosine is 0, and cotangent and cosecant are analytic except where sine is 0.

We define the hyperbolic trig functions:

$$\begin{aligned} \cosh z &= \frac{1}{2}\left(e^z + e^{-z}\right) \\ \sinh z &= \frac{1}{2}\left(e^z - e^{-z}\right) \end{aligned}$$

These are both entire functions.

1.2 Branch Cuts

So far all the generalizations have been pretty well-behaved, but now we hit the first snag, when we consider roots. Consider $w^n = z$. Then $w = z^{\frac{1}{n}}$. We now show that for any non-zero z, there are n distinct values of w. In polar coordinates, we have that

$$z = re^{i\theta}$$

and

$$w = Re^{i\phi}$$

Now exponentiating w in the polar form:

$$\begin{aligned} w^n &= R^n e^{in\phi} \\ &= z \\ &= re^{i\theta} \end{aligned}$$

Thus we have that $R^n = r$, and therefore R is the nth root of r, $R = r^{\frac{1}{n}}$. Also note that since both r and R are real and positive, R is unique.

Now let us consider the phase terms, $e^{in\phi}$ and $e^{i\theta}$. The condition for these two to be equal is:

$$n\phi = \theta + 2\pi k$$

for any $k \in \mathbb{Z}$. Thus we have that

$$\phi = \frac{\theta}{n} + \left(\frac{k}{n}\right) 2\pi$$

We see that ϕ takes on n values, with $k = 0, 1, \ldots, n-1$. Thus we see that when asking for the nth root of z, there are n different valid roots. Why can we not just pick a principal root, such as $k = 0$? Suppose we have $\theta \in [0, 2\pi)$, and picking $k = 0$. Then we have forced the nth root of z to be single-valued. Now consider the value of $z^{\frac{1}{n}}$ at a point just above the real axis. In this case, $\theta \approx 0$, and therefore $z^{\frac{1}{n}} = r^{\frac{1}{n}}$. We can trace out a circle around the origin, and the value smoothly changes as θ changes. However, as we approach the real axis from below, we see that the value of the function approaches $r^{\frac{1}{n}} e^{i\frac{2\pi}{n}}$. We see that we have a sudden jump across the real axis. A region in which the function's value jumps suddenly like this is known as a **branch point**. In this case the branch point is the positive real axis.

In fact, the branch cut singularity is unavoidable, we can shift the choice of k, and we will just shift the branch cut. For example, suppose we restrict θ to lie between $-\pi$ and π. In this case, the branch cut will occur on the negative real axis, where θ wraps back around.

Let us now consider the logarithm. Suppose $e^w = z$, and thus $w = \ln z$. If we write z in polar form, $z = re^{i\theta}$:

$$\begin{aligned} \ln z &= \ln\left(re^{i\theta}\right) \\ &= \ln r + \ln\left(e^{i\theta}\right) \\ &= \ln r + i\left(\theta + 2\pi k\right) \end{aligned}$$

where $k \in \mathbb{Z}$ is an arbitrary shift to the angle. We see that, in general, $\ln z$ is multi-valued. We can make it single valued by restricting the choice of k to a single value, such as $k = 0$, but then we have a branch cut singularity.

To show this, let us restrict ourselves to $k = 0$ and $-\pi < \theta \leq \pi$. We see that if we look at the negative real axis, we find that approaching from above gives us $\theta \to \pi$, and from below gives us $\theta \to -\pi$. Thus $w = \ln r + i\pi$ from above, and $w = \ln r - i\pi$ from below, which does not match, and thus we have a branch cut singularity.

We can also consider other functions, in which the singularity structure may be more complicated. Consider the seemingly simple function

$$\begin{aligned} f(z) &= \left(z^2 - 1\right)^{1/2} \\ &= (z+1)^{1/2} (z-1)^{1/2} \end{aligned}$$

We can see that we expect a branch cut starting at $z = 1$, as well as a branch cut starting at $z = -1$. What is the singularity structure of the product of these two functions?

We claim that one of the allowed choices of a branch cut is the line between $z = -1$ and $z = 1$. Let us determine whether or not this is true. First, let us write the two in polar form:

$$\begin{aligned} (z+1) &= re^{i\theta} \\ (z-1) &= \rho e^{i\phi} \end{aligned}$$

Suppose we choose the range $\theta \in [0, 2\pi)$ and $\phi \in [0, 2\pi)$. Now, given these definitions, $f(z) = \sqrt{\rho r} e^{i(\theta+\phi)/2}$. Let us track the value of the phase component as we move along a path that goes around our two points (and doesn't intersect where we claim the branch cut is). We are looking for sudden jumps in the value of the phase, which would indicate a branch cut.

We can obtain the values of θ by measuring the angle from the point $z = -1$, and obtain the value of ϕ by measuring the angle from the point $z = +1$. After doing so, we can compute the phase on the product. We then look for sharp discontinuities when transitioning from above to below the axis, or vice versa. We see that we have no discontinuities past $z = 1$ or before $z = -1$, and we have discontinuities when transitioning between the two. This validates our choice of a branch cut lying between $z = -1$ and $z = +1$.

What if we make a different choice of our restrictions? Can we change the location of the branch cut?

Suppose we choose $\theta \in [-\pi, \pi]$, and leave the ϕ restriction the same. First, are we allowed to pick the ranges of the two angles to be different from each other? To show that we can, suppose we write the two subfunctions as follows:

$$\begin{aligned} \sqrt{z-1} &= R_- e^{i\theta_-} \\ \sqrt{z+1} &= R_+ e^{i\theta_+} \end{aligned}$$

And we can square both of these:

$$z - 1 = R_-^2 e^{2i\theta_-}$$

$$z + 1 = R_+^2 e^{2i\theta_+}$$

Note that we can consistently change θ_- by π, and both formulas are still valid. Thus, we are allowed to choose the phases arbitrarily. This means that we can change the cut structure by making different choices for the ranges of θ and ϕ. If we return to the choice where $\theta \in [-\pi, \pi)$, we will find that we have branch cuts before $z = -1$ and ater $z = +1$, but no branch cut in between.

1.3 Line Integrals

Consider a line integral of a function $f(z)$ from a point A to a point B in the complex plane. Under what circumstances is the value of this integral $\int_A^B f(z)\, dz$ independent of the path taken from A to B.

Theorem 1.4. *Cauchy's Theorem. If $f(z)$ is analytic in a simply connected, bounded domain D. For every simple closed path C in D:*

$$\oint_C f(z)\, dz = 0$$

Where $\oint$ denotes the fact that C is a closed path. A simply connected domain implies that any closed curve in the domain can be shrunk to a point without leaving the domain D.

Proof. We can write out the integral:

$$\begin{aligned}\oint_C f(z)\, dz &= \oint_C (u + iv)(dx + i dy) \\ &= \oint_C (u\, dx - v\, dy) + i \oint_C (v\, dx + u\, dy)\end{aligned}$$

Now applying Stoke's theorem:

$$\oint \boldsymbol{A} \cdot d\boldsymbol{r} = \int (\nabla \times \boldsymbol{A}) \cdot d\boldsymbol{s}$$

Let us attempt to map the line integrals that we have into the form of Stoke's theorem. In the plane, $d\boldsymbol{r} = dx\hat{x} + dy\hat{y}$, and $\boldsymbol{A} = A_x\hat{x} + A_y\hat{y}$. Rewriting Stoke's theorem:

$$\oint_C (A_x\, dx + A_y\, dy) = \int \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) dx\, dy$$

Now let us compare the left side of this equation to the line integrals that we have. We see that if we choose $A_x = u$ and $A_y = -v$, we have a perfect mapping for the real line integral. On the right side of Stoke's theorem, we will have:

$$\oint_C (u\, dx - v\, dy) = \int \underbrace{\left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)}_{0} dx\, dy$$

Now we note that by the Cauchy-Riemann conditions (1.1), this right integral is 0, and therefore the line integral is also 0. For the imaginary line integral, we see that again, by the Cauchy-Riemann conditions (1.1), Stoke's theorem tells us that the integral will be 0. Thus we have that both line integrals will be 0, and thus the integral over the closed contour C is 0. Note that the requirement that the domain be simply connected is in place because Stoke's theorem requires the existence of the partial derivatives of u and v. □

We wanted to find out whether the line integral between two points is independent of the path chosen:

$$\int_{C_1} f(z)\, dz = \int_{C_2} f(z)\, dz$$

This is implied by Cauchy's theorem, since we can form a closed path by following C_1 to B, and then the inverse of C_2 back to A. The integral over this closed contour must be 0, and therefore the two integrals must be equal and opposite. Thus the original line integrals must be equivalent.

Because of Cauchy's theorem , we can actually treat integrals as functions:

$$\int_a^z f(z)\, dz = F(z)$$

This is a function of only z, since it is independent of the path that we choose from point a to z.

1.4 Cauchy's Integral Formula

Cauchy's integral formula is a theorem that allows us to obtain the value of an analytic function anywhere inside a simply connected domain, as long as we know its value everywhere on the boundary of the domain. This is incredibly powerful, and is due to the fact that $f(z) = u + iv$, where u and v are both solutions of Laplace's equation.

Theorem 1.5. *Cauchy's Integral Formula. If $f(z)$ is analytic in a simply connected domain D, then for any point $z = a$ in D and any closed path C in D which encloses the point a:*

$$\oint_C \frac{f(z)}{z-a}\, dz = 2\pi i f(a)$$

Where the integration is taken in the counter-clockwise sense.

Proof. Let us define $\phi(z) = \frac{f(z)}{z-a}$. This is analytic everywhere in D except $z = a$. What we want to evaluate is:

$$\oint \phi(z)\, dz$$

This does not immediately integrate to 0 because of the fact that the function is not analytic at $z = a$. Consider a new contour inside C, C', which is an arbitrarily small circle around $z = a$. We now introduce a "cut", that joins C and C'. This closed contour essentially cuts out $z = a$, it contains all the points other than $z = a$. Thus, $\phi(z)$ is completely analytic in this region. Note that the cut goes around $z = a$ in a clockwise manner.

For this new contour,

$$\oint_{C_{\text{new}}} \phi(z)\, dz = 0$$

by Cauchy's theorem (1.4). Now we note that the net contributions of the cuts that connect C to C' vanish as we make them infinitely close to each other. What we are left with is the contribution from the outer contour and the contribution from the circle around $z = a$:

$$\oint_C \phi(z)\, dz + \oint_{C'} \phi(z)\, dz = 0$$

Note that the integral around C is counterclockwise, and the integral around C' is clockwise. Thus we have that

$$\oint_C \phi(z)\, dz = \oint_{C'} \phi(z)\, dz$$

Where we have now made the integral around C' go counterclockwise, which cancels out the negative that is introduced. Now let us attempt to compute the right integral. The equation for a circle centered at $z = a$ of radius ρ is given by

$$z - a = \rho e^{i\theta}$$

And thus $|z - a| = \rho$. We can compute the integral:

$$\begin{aligned}\oint_{C'} \phi(z)\, dz &= \oint_{C'} \frac{f(z)}{z-a}\, dz \\ &= \int_0^{2\pi} \frac{f(z)}{\rho e^{i\theta}} \left(i\rho e^{i\theta}\, d\theta\right) \\ &= i\int_0^{2\pi} f(z)\, d\theta\end{aligned}$$

Now we take the radius of the circle to be infinitely small, which allows us to replace $f(z)$ with $f(a)$:

$$\oint_C \frac{f(z)}{z-a}\, dz = 2\pi i f(a)$$

□

1.5 Laurent Series

Theorem 1.6. *Laurent's Theorem. Let C_1 and C_2 be two circles centered at z_0. Let $f(z)$ be analytic in the region R between the circles. Then $f(z)$ can be expanded in a series of the form (denoted a Laurent Series)*

$$f(z) = \underbrace{a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \ldots}_{\text{Taylor Series}} + \overbrace{\frac{b_1}{(z-z_0)} + \frac{b_2}{(z-z_0)^2} + \ldots}^{\text{Principal Part}}$$

This series converges and represents $f(z)$ in the open annulus obtained from the given annulus by continuously increasing the circle C_1 (the outer boundary) and decreasing C_2 (the inner boundary) until each of the two circles reaches a point where $f(z)$ is singular.

Essentially, this is a Taylor series that contains the negative powers as well. We can increase the size of the annulus as much as we want, as long as neither of the boundaries pass through a singular point.

Note that if $f(z)$ is analytic at $z = z_0$, we find that in the region that contains z_0, the principal part of the series vanishes.

We also state a related property:

$$\oint \frac{dz}{(z-z_0)^n} = \begin{cases} 2\pi i & \text{if } n = 1 \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

Using this result, we can determine the coefficients of the series:

$$a_n = \frac{1}{2\pi i} \oint \frac{f(z)}{(z-z_0)^{n+1}}\, dz$$
$$b_n = \frac{1}{2\pi i} \oint \frac{f(z)}{(z-z_0)^{-n+1}}\, dz$$

This is generally not the way that the Laurent series is found in practice.

Let us do a couple of examples. Consider the function $f(z) = z^2 e^{1/z}$. Find the Laurent series of $f(z)$ centered at $z = 0$.

First we note that the function is not analytic at $z = 0$, due to the $e^{1/z}$, and thus $z = 0$ is a sinular point. This implies that the principal part of the Laurent series will not vanish.

Using the fact that

$$e^z = 1 + z + \frac{z^2}{2} + \frac{z^3}{3!} + \dots$$

We can write out $f(z)$:

$$\begin{aligned} z^2 e^{1/z} &= z^2 \left[1 + \frac{1}{z} + \frac{1}{2}\frac{1}{z^2} + \frac{1}{3!}\frac{1}{z^3} + \dots \right] \\ &= z^2 + z + \frac{1}{2} + \frac{1}{3!}\frac{1}{z} + \dots \end{aligned}$$

This converges for $\forall z$ such that $|z| > 0$, everywhere except for the point we are expanding about.

Let us do another example. What is the Laurent series of $f(z) = \frac{1}{1-z^2}$ centered at $z = 1$?

Immediately we see that the function has singularities at $z = \pm 1$, and if we are centered at $z = 1$, we have 2 Laurent series. We have a circle centered at $z = 1$, and it passes through $z = -1$. Inside the circle we have 1 Laurent series, and outside the circle we have another. We can rewrite $f(z)$:

$$\begin{aligned} \frac{1}{1-z^2} &= \frac{1}{1-z}\frac{1}{1+z} \\ &= -\frac{1}{z-1}\frac{1}{z+1} \end{aligned}$$

Now we note that we are looking for a series in powers of $z - 1$, so all we have to do is to expand $\frac{1}{z+1}$ in powers of $z - 1$:

$$\begin{aligned} \frac{1}{z+1} &= \frac{1}{2+(z-1)} \\ &= \frac{1}{2}\left[\frac{1}{1+\frac{z-1}{2}} \right] \end{aligned}$$

We do this so we can exploit the binomial expansion:

$$\frac{1}{(1+x)^n} = 1 - n + \frac{n(n+1)}{2}x^2 + \dots$$

Which has the condition that $|x| < 1$. Applying this, in the case where $\left|\frac{z-1}{2}\right| < 1$, which is the same as $|z - z| < 2$, we can expand binomially:

$$\begin{aligned}\frac{1}{z+1} &= \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z-1}{2}\right)^n \\ &= \sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}(z-1)^n\end{aligned}$$

Now tacking on the $-\frac{1}{z-1}$ to recover $f(z)$:

$$\begin{aligned}\frac{1}{1-z^2} &= -\frac{1}{z-1}\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}(z-1)^n \\ &= \sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{2^{n+1}}(z-1)^{n-1}\end{aligned}$$

This is the the Laurent series inside the circle (because the binomial expansion happens to impose that condition). To find the Laurent series outside the circle, we return to the rewritten form of $\frac{1}{1+z}$:

$$\begin{aligned}\frac{1}{1+z} &= \frac{1}{2}\left[\frac{1}{1+\left(\frac{z-1}{2}\right)}\right] \\ &= \frac{1}{z-1}\left[\frac{1}{1+\frac{2}{z-1}}\right]\end{aligned}$$

We see that we can expand out the second term binomially, which imposes the condition that $|z-1| > 2$, which is exactly the condition that we want, z that is outside of the circle.

$$\begin{aligned}\frac{1}{1+z} &= \frac{1}{z-1}\sum_{n=0}^{\infty}(-1)^n\left(\frac{2}{z-1}\right)^n \\ &= \sum_{n=0}^{\infty}(-1)^n\frac{2^n}{(z-1)^{n+1}}\end{aligned}$$

We can now add back in the $\frac{1}{z-1}$:

$$\begin{aligned}f(z) &= -\frac{1}{z-1}\sum_{n=0}^{\infty}(-1)^n\frac{2^n}{(z-1)^{n+1}} \\ &= \sum_{n=0}^{\infty}(-1)^{n+1}\frac{2^n}{(z-1)^{n+2}}\end{aligned}$$

Giving us the second Laurent series for $f(z)$.

Let us now look at another function. Consider $f(z) = \frac{12}{z(2-z)(1+z)}$. We want to find all of the Laurent series of this function, centered at $z = 0$.

To begin, we find the singular points of $f(z)$. We have 3 singular points, $z = 0, 2, -1$. We expect to have 3 Laurent series, one valid in the regime where $|z| < 1$, another between $1 < |z| < 2$, and a third outside, $|z| > 2$.

We can rewrite the function:

$$\begin{aligned} f(z) &= \frac{12}{z(2-z)(1+z)} \\ &= \frac{1}{z}\left[\frac{12}{(2-z)(1+z)}\right] \end{aligned}$$

We want to expand the contents of the brackets in terms of powers of z, since we are centered at $z = 0$. To do this, we use partial fraction decomposition:

$$\frac{12}{(2-z)(1+z)} = 4\left[\frac{1}{2-z} + \frac{1}{1+z}\right]$$

Now we can expand each of the two separate partial fractions as binomial series. In the case where $|z| < 1$, we can write out $1/(1+z)$:

$$\frac{1}{1+z} = \sum_{n=0}^{\infty} (-1)^n z^n$$

And we can write out $1/(2-z)$:

$$\begin{aligned} \frac{1}{2-z} &= \frac{1}{2}\frac{1}{1-\frac{z}{2}} \\ &= \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n \\ &= \sum_{n=0}^{\infty}\frac{z^n}{2^{n+1}} \end{aligned}$$

Thus, for $|z| < 1$:

$$\begin{aligned} f(z) &= \frac{4}{z}\left[\sum_{n=0}^{\infty}(-1)^n z^n + \sum_{n=0}^{\infty}\frac{z^n}{2^{n+1}}\right] \\ &= \end{aligned}$$

Now let us consider the outermost region, where $|z| > 2$. We can once again binomially expand:

$$\begin{aligned} \frac{1}{1+z} &= \frac{1}{z}\left[\frac{1}{1+\frac{1}{z}}\right] \\ &= \frac{1}{z}\sum_{n=0}^{\infty}(-1)^n\frac{1}{z^n} \end{aligned}$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{1}{z^{n+1}}$$

And we can expand $1/(2-z)$:

$$\begin{aligned} \frac{1}{2-z} &= \frac{1}{z}\left[\frac{-1}{1-\frac{2}{z}}\right] \\ &= -\frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{2}{z}\right)^n \\ &= \sum_{n=0}^{\infty} -\frac{2^n}{z^{n+1}} \end{aligned}$$

Putting these together:

$$\begin{aligned} f(z) &= \frac{4}{z}\left[\sum_{n=0}^{\infty}(-1)^n\frac{1}{z^{n+1}} + \sum_{n=0}^{\infty} -\frac{2^n}{z^{n+1}}\right] \\ &= \frac{4}{z}\left[\sum_{n=0}^{\infty} -\left[-2^n-(-1)^n\right]\frac{1}{z^{n+1}}\right] \end{aligned}$$

Now we only have the intermediate region left. We did it in this particular order because we can use one expansion from each of the two regions that we have already done (based on the valid region of each expansion). For $1 < |z| < 2$, we have

$$\begin{aligned} \frac{1}{1+z} &= \sum_{n=0}^{\infty}(-1)^n\frac{1}{z^{n+1}} \\ \frac{1}{2-z} &= \sum_{n=0}^{\infty}\frac{z^n}{2^{n+1}} \end{aligned}$$

Putting these two together:

$$f(z) = \frac{4}{z}\left[\sum_{n=0}^{\infty}(-1)^n\frac{1}{z^{n+1}} + \sum_{n=0}^{\infty}\frac{z^n}{2^{n+1}}\right]$$

Thus we have found all 3 Laurent series for the function.

By looking at the principal part of the Laurent series, we can state some more definitions. If a single term in the principal part is nonzero, and the rest are zero, suppose the order nth term, then $f(z)$ is said to have a *pole* of order n at z_0. If $n = 1$, it is denoted a *simple pole*.

If all of the b_ns vanish, then $f(z)$ is analytic at z_0. If an infinite number of the b_ns are nonzero, then we say that $f(z)$ has an *essential singularity* at $z = z_0$.

The coefficient b_1 is known as the **residue** of $f(z)$ at z_0.

Theorem 1.7. *Liouville's Theorem. If $f(z)$ is analytic and bounded in absolute value for all z in the complex plane, then it must be a constant.*

Before we begin the proof, we introduce some statements that are useful:

$$|ab| = |a||b|$$

To prove this statement, we note that $a = |a|e^{i\theta}$, and $b = |b|e^{i\phi}$. From this, we immediately find that

$$ab = |a||b|e^{i(\theta+\phi)}$$

And therefore $|ab| = |a||b|$.

The second useful property is that

$$|a + b| \leq |a| + |b|$$

Proof. Assume that $|f(z)|$ is bounded, so that $|f(z)| < M$ for all z. Since $f(z)$ is analytic, by Laurent's theorem (1.6) it can be expanded: $f(z) = \sum_{n=0}^{\infty} a_n z^n$. Note that we don't have negative powers of n because the function is analytic everywhere. Now using (3):

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z^{n+1}}\, dz$$

The contour in this case is a circle of radius R centered at the origin. Now we take the absolute value of both sides:

$$\begin{aligned} |a_n| &= \left| \frac{1}{2\pi i} \oint_C \frac{f(z)}{z^{n+1}}\, dz \right| \\ &= \left| \frac{1}{2\pi i} \right| \left| \oint_C \frac{f(z)}{z^{n+1}}\, dz \right| \end{aligned}$$

Now imagine we break up the integral into an infinite sum over the path elements m:

$$|a_n| = \left| \frac{1}{2\pi i} \right| \left| \sum_m \frac{f(z_m)}{z_m^{n+1}} \Delta z_m \right|$$

Now noting that $|a + b| \leq |a| + |b|$, we have that (working in the limit where $m \to 0$)

$$\begin{aligned} |a_n| &\leq \frac{1}{2\pi} \left[\sum_m \left| \frac{f(z_m)}{z_m^{n+1}} \Delta z_m \right| \right] \\ &\leq \frac{1}{2\pi} \sum_m \frac{|f(z_m)|}{|z_m|^{n+1}} |\Delta z_m| \end{aligned}$$

Recall that we are travelling around a circle of radius R, so at every point on the contour, $|z_m| = R$.

$$|a_n| \leq \frac{1}{2\pi} \sum_m \frac{|f(z_m)|}{R^{n+1}} |\Delta z_m|$$

As we move around the circle, at some point, $f(z_m)$ will take on a maximum value. Thus this statement must be less than the case where every point is the maximum value:

$$|a_n| \leq \frac{1}{2\pi} \sum_m \frac{|f(z)|_{\max}}{R^{n+1}} |\Delta z_m|$$

$$\leq \frac{1}{2\pi} \frac{|f(z)|_{\max}}{R^{n+1}} \sum_m |\Delta z_m|$$

Now noting that $\sum_m |\Delta z_m| = 2\pi R$:

$$\begin{aligned} |a_n| &\leq \frac{1}{2\pi} \frac{|f(z)|_{\max}}{R^{n+1}} (2\pi R) \\ &\leq \frac{|f(z)|_{\max}}{R^n} \end{aligned}$$

Now since we assumed that $f(z)$ was bounded by M, $|f(z)|_{\max}$ must be less than M:

$$|a_n| \leq \frac{M}{R^n}$$

Now consider taking the limit as $R \to \infty$. We see that this forces $|a_n|$ to 0 for $n > 0$, and since the coefficients of the expansion other than a_0 are 0, the function must be constant. □

1.6 Residue Theorem

Theorem 1.8. ***Residue Theorem.*** *Let z_0 be an isolated singular point of $f(z)$. Consider the value of $\oint_C f(z)\, dz$ around a simple closed curve C surrounding z_0, but enclosing no other singularities. Let $f(z)$ be expanded in a Laurent series about $z = z_0$ so that it converges near $z = z_0$.*

$$f(z) = a_0 + a_1(z - z_0) + \cdots + \frac{b_1}{z - z_0} + \frac{b_2}{(z - z_0)^2} + \dots$$

The terms of the "a series" do not contribute to the integral, because they are analytic by Cauchy's theorem (1.4). By (3), the only one of the terms in the "b series" that contributes is the order 1 term:

$$\oint_C f(z)\, dz = 2\pi i b_1$$

Now consider the situation where the contour contains several isolated singularities, not just one. In this case, we can draw contours around each singularity, and then use the same trick that we did for Cauchy's Integral Formula, and we introduce slices that merge all of the contours. The contributions of the slices cancel. We are left with the sum of all of the residues inside the region:

$$\oint_C f(z)\, dz = 2\pi i \,[\text{sum of residues inside } C]$$

Let's do an example. Consider the function $f(z) = \frac{\sin z}{z^4}$. We want to integrate this function around the unit circle going counter-clockwise.

We can first write this as a Laurent series around the origin:

$$\begin{aligned} \frac{\sin z}{z^4} &= \frac{1}{z^4}\left[z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots\right] \\ &= \frac{1}{z^3} - \frac{1}{6}\frac{1}{z} + \frac{z}{5!} + \dots \end{aligned}$$

The only singularity of $f(z)$ is at $z = 0$, and the residue is $-\frac{1}{6}$. By the residue theorem (1.8):

$$\begin{aligned}\oint_C \frac{\sin z}{z^4}\,dz &= 2\pi i\left(-\frac{1}{6}\right)\\ &= -\frac{\pi i}{3}\end{aligned}$$

Now let us consider the function $f(z) = \frac{4-3z}{z^2-z}$ around the circle $|z| = 2$.

The function has singularities at $z = 0$ and $z = 1$. First, we note that both singularities are inside the contour, and thus are both important.

Near $z = 0$:

$$\begin{aligned}f(z) &= \frac{4-3z}{z(z-1)}\\ &\approx -\frac{4}{z} + \text{smooth}\end{aligned}$$

From this, we have that the residue of $f(z)$ near $z = 0$ is -4.

Similarly, near $z = 1$:

$$\begin{aligned}f(z) &= \frac{4-3z}{z(z-1)}\\ &\approx \frac{+1}{z-1} + \text{smooth}\end{aligned}$$

And we see that the residue of $f(z)$ near $z = 1$ is $+1$.

Now applying the residue theorem (1.8):

$$\begin{aligned}\oint \frac{4-3z}{z^2-z}\,dz &= 2\pi i\,[-4+1]\\ &= -6\pi i\end{aligned}$$

Now consider the integral:

$$I = \int_0^{2\pi} \frac{1}{5+4\cos\theta}\,d\theta$$

This integral requires a difficult substitution ($\tan\frac{\theta}{2}$), so instead, let us change variables to a complex variable, $z = e^{i\theta}$. When θ goes from 0 to 2π, our angle goes around the unit circle. Computing the change of variables:

$$\begin{aligned}dz &= ie^{i\theta}\\ &= iz\,d\theta\end{aligned}$$

From this, we have that $d\theta = \frac{1}{i}\frac{dz}{z}$. Using Euler's formula, we have that

$$\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right)$$

Now rewriting the integral:

$$\begin{aligned} I &= \oint_{\text{unit circle}} \frac{1}{5+2\left(z+\frac{1}{z}\right)} \frac{1}{iz}\, dz \\ &= \frac{1}{i} \oint \frac{1}{2z^2+5z+2} \\ &= \frac{1}{i} \oint \frac{dz}{(2z+1)(z+2)} \end{aligned}$$

This function has poles at $z=-2$ and $z=-\frac{1}{2}$. We note the pole at $z=-2$ does not contribute, it is outside the contour. Thus, by the residue theorem (1.8), we just need the residue at $z=-\frac{1}{2}$, so we consider the function near $-\frac{1}{2}$:

$$\begin{aligned} f\left(z \to -\frac{1}{2}\right) &= \frac{1}{(2z+1)(z+2)} \\ &\approx \frac{\frac{1}{3}}{\left(z+\frac{1}{2}\right)} \end{aligned}$$

Thus we have a residue of $\frac{1}{3}$. By the residue theorem (1.8):

$$\begin{aligned} I &= \left(\frac{1}{i}\right) 2\pi i \left(\frac{1}{3}\right) \\ &= \frac{2\pi}{3} \end{aligned}$$

Let's do another example. Consider the integral:

$$I = \int_{-\infty}^{\infty} \frac{dx}{1+x^2}$$

This is a standard integral, $I = \arctan(x)\,|_{-\infty}^{\infty} = \pi$. However, let us do this for illustrative purposes.

How do we map this integral into the complex plane? Consider the complex line integral

$$\lim_{\rho\to\infty} \int_{-\rho}^{\rho} \frac{1}{1+z^2}$$

Where the path is along the real line, so the two integrals are identical. Note that this is not a closed line integral, and therefore we cannot apply the residue theorem directly.

Now consider a circle of radius ρ, centered at the origin. The points of this circle that lie on the real axis are $z=-\rho$ and $z=\rho$. Consider the closed contour that starts at $-\rho$, travels along the real line, and then follows the upper semicircle to end up at $-\rho$.

Now let us integrate the function around this new closed contour:

$$\oint \frac{dz}{1+z^2}$$

Note that we can now apply the residue theorem.

The point of this is to note that the contour integral is made up of two parts, the real axis contributions, and the contributions from the semicircle. What we will show is that the contributions

of the integral along the semicircle vanish, and we will be left with just the integral along the real axis, which is what we wanted.

The function $\frac{1}{1+z^2}$ has simple poles at $z = \pm i$. Only one of these poles is in the contour that we care about, $z = +i$. Near $z = +i$:

$$\begin{aligned} f(z \to +i) &= \frac{1}{1+z^2} \\ &\approx \frac{1}{2i}\frac{1}{z-i} \end{aligned}$$

And so we have a residue of $\frac{1}{2i}$. By the residue theorem (1.8):

$$\begin{aligned} \oint \frac{dz}{1+z^2} &= 2\pi i \left(\frac{1}{2i}\right) \\ &= \pi \end{aligned}$$

Now we need to argue that the contributions on the semicircle vanish. We can separate out the integral:

$$\underbrace{\oint \frac{dz}{1+z^2}}_{\pi} = \underbrace{\int_{\text{real}} \frac{dz}{1+z^2}}_{I} + \int_{\text{semicircle}} \frac{dz}{1+z^2}$$

To evaluate the contributions on the semicircle, consider $z = \rho e^{i\theta}$, with constant ρ, and $\theta = 0 \to \pi$. In this case, $dz = i\rho e^{i\theta}\, d\theta = iz\, d\theta$. This turns the semicircle integral into:

$$\int_{\text{semicircle}} \frac{dz}{1+z^2} = \int_0^\pi \frac{i\rho e^{i\theta}\, d\theta}{1+\rho^2 e^{2i\theta}}$$

Now consider the absolute value of both sides:

$$\left| \int_{\text{semicircle}} \frac{dz}{1+z^2} \right| = \left| \int_0^\pi \frac{i\rho e^{i\theta}\, d\theta}{1+\rho^2 e^{2i\theta}} \right|$$

Looking at the right side, and using similar logic as in our proof of Liouville's theorem:

$$\left| \int_0^\pi \frac{i\rho e^{i\theta}\, d\theta}{1+\rho^2 e^{2i\theta}} \right| \leq \int_0^\pi \frac{|i\rho e^{i\theta}|}{|1+\rho^2 e^{2i\theta}|} |d\theta|$$

As $\rho \to \infty$, the integrand scales like $\frac{1}{\rho}$:

$$\begin{aligned} \left| \int_0^\pi \frac{i\rho e^{i\theta}\, d\theta}{1+\rho^2 e^{2i\theta}} \right| &\leq \frac{1}{\rho}\pi \\ &\to 0 \text{ as } \rho \to \infty \end{aligned}$$

And thus the contribution of the integral along the semicircle vanish as we approach the infinite bounds.

Let us consider another example. Suppose we want to evaluate the integral

$$I = \int_0^\infty \frac{\cos x}{1+x^2}\, dx$$

First, we note that the integrand is even, so we can write it as half of the integral over the entire real line:

$$I = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\cos x}{1+x^2}\,dx$$

Now using the same technique as the previous problem:

$$I = \frac{1}{2}\int_{C} \frac{\cos z}{1+z^2}\,dz$$

Where C is the entire real line. Now we note that we can rewrite the integral as the real part of a complex integral:

$$I = \text{Re}\left[\frac{1}{2}\int_{C} \frac{e^{iz}}{1+z^2}\,dz\right]$$

Now we create a closed contour like we did in the previous example, creating a semicircle in the upper half of the complex plane. We now are evaluating the integral:

$$\frac{1}{2}\oint \frac{e^{iz}}{1+z^2}\,dz$$

We first find the singularities of this function, which are $z = \pm i$. Only one of these poles is in the contour, $z = +i$. We can compute the residue at that pole, by seeing what happens near $z = +i$:

$$\frac{1}{2}\frac{e^{-1}}{2i\,(z-i)}$$

Which gives us residue $\frac{1}{4ie}$. This gives us, by the Residue Theorem (1.8):

$$\frac{1}{2}\oint \frac{e^{iz}}{1+z^2}\,dz = \frac{\pi}{2e}$$

Now we have to show that the contribution of the semicircle contour vanishes. Looking at the integrand, suppose we have $z = \rho e^{i\theta} = \rho\cos\theta + \rho\sin\theta$, where $\rho \to \infty$. We can rewrite the integrand:

$$\frac{e^{iz}}{1+z^2} = \frac{e^{i\rho\cos\theta}e^{-\rho\sin\theta}}{1+z^2}$$

In the entire half plane, θ is between 0 and π, and $\sin\theta$ is therefore positive, so $e^{-\rho\sin\theta} \to 0$ as $\rho \to \infty$. Thus we have that the integral of the semicircle contour vanishes.

Note that if we looked at the lower half plane, $\sin\theta$ would be negative, and thus the contour would blow up. If we really wanted to use the lower half plane, we could have written the integral as the real part of the integral of e^{-iz}, which produces the same result but would cause the lower semicircle contour to vanish.

Now let us consider another example. Suppose we have the integral

$$I = \int_{0}^{\infty} \frac{r^{p-1}}{1+r}\,dr$$

Where $p \in (0, 1)$.

Turning this into a complex integral:

$$I = \int_C \frac{z^{p-1}}{1+z}\, dz$$

Where C is the positive real line. Note that since we have a fractional power of z, there is no way of having this be single-valued. We write $z = Re^{i\theta}$, so $z^{p-1} = R^{p-1}e^{i\theta(p-1)}$. If we go around $\theta \to \theta + 2\pi$, z^{p-1} becomes $R^{p-1}e^{i\theta(p-1)}e^{i2\pi(p-1)}$. This is the expected phase that gives us a multi-valued function. z^{p-1} has a branch cut, which we take along the positive x axis.

We are integrating along $\theta = 0$, the positive real axis. Now we want to choose the contour that we are integrating along. We choose the contour that integrates around everything except the branch cut (a Pacman shape that avoids the branch cut). We will show that the contribution from the circle vanishes, and the contribution from the integral we want, and neither does its analogue below the branch cut.

We can write out the integral along the circle:

$$\int_{\text{circle}} \frac{R^{p-1}e^{i\theta(p-1)}}{1+Re^{i\theta}} Re^{i\theta}\, (i\, d\theta)$$

Now if we take the absolute value of the integrand:

$$\int_{\text{circle}} \left| \frac{R^{p-1}e^{i\theta(p-1)}}{1+Re^{i\theta}} Re^{i\theta} \right| (i\, d\theta)$$

We see that $\left|R^{p-1}\right| \to 0$ as $R \to \infty$, and thus the integral over the circle contour vanishes.

Now let us consider the section of the contour that we are trying to evaluate. We have that $\theta = 0$, and we have the integral:

$$\int_0^\infty \frac{r^{p-1}}{1+r}\, dr$$

Along the contour from ∞ to 0 along the real axis from underneath the branch cut, we have that $\theta = 2\pi$, so $z = re^{i(2\pi)}$, so this gives the integral:

$$\begin{aligned}\int_\infty^0 \frac{\left(re^{2\pi i}\right)^{p-1}}{1+r}\, dr &= -\int_0^\infty \frac{r^{p-1}e^{2\pi ip}}{1+r} \\ &= -e^{2\pi ip}\int_0^\infty \frac{r^{p-1}}{1+r}\, dr\end{aligned}$$

The total contribution from the two sections along the branch cut is given by

$$\left[1 - e^{2\pi ip}\right] \underbrace{\int_0^\infty \frac{r^{p-1}}{1+r}\, dr}_{I}$$

Where we see that we can extract the value of the integral that we are trying to compute.

Since the whole thing is a closed contour, we can evaluate the total integral using the residue theorem, and then solve for the value of I.

We want to compute

$$\oint \frac{z^{p-1}}{1+z}\,dz$$

We see that we have a simple pole at $z=-1$, which is inside the contour that we are integrating around. We can rewrite $z=-1$ as $e^{i\pi}$. Computing the residue around $z=-1$, we see that

$$\frac{z^{p-1}}{1+z} \to \frac{e^{i\pi(p-1)}}{1+z}$$

So we have a residue of $e^{i\pi(p-1)}$. By the Residue Theorem (1.8):

$$\oint \frac{z^{p-1}}{1+z}\,dz = -2\pi i e^{i\pi p}$$

Thus we have that

$$-2\pi i e^{i\pi p} = \left[1-e^{2\pi i p}\right] \underbrace{\int_0^\infty \frac{r^{p-1}}{1+r}\,dr}_{I}$$

$$I = \frac{\pi}{\sin(\pi p)}$$

Consider the integral:

$$I = \int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x}\,dx$$

For $a \in (0,1)$.

If we change x to z, the denominator would be $1+e^z$. Let us consider what the poles would be. We would have poles at $z = \pm i\pi, \pi i\,(3\pi), \pm i\,(5\pi), \ldots$. We see that we have an infinite number of poles. To avoid this problem, we choose the contour that forms a rectangle in the upper complex plane, with height 2π. The base and top of the rectangle extend out from $-\infty$ to ∞.

We will show that the contribution from the sides vanishes. The left side ranges from $z=(-R,0)$ to $z=(-R,2\pi i)$, and the right side ranges from $z=(R,0)$ to $z=(R,2\pi i)$, and we take the limit as $R\to\infty$. We can split up the contour integral that we claim remains:

$$\oint \frac{e^{az}}{1+e^z}\,dz = \lim_{R\to\infty}\left[\underbrace{\int_{-R}^{R}\frac{e^{ax}}{1+e^x}\,dx}_{\text{Real Axis}} - \underbrace{\int_{-R}^{R}\frac{e^{ax}}{1+e^x}e^{2\pi i a}\,dx}_{\text{Top Contour}}\right]$$

Where the $e^{2\pi i a}$ is the overall phase that is picked up on the upper line parallel to the real axis.

Pulling the phase out, we have that

$$\oint \frac{e^{az}}{1+e^z}\,dz = \left(1-e^{2\pi i a}\right)\underbrace{\int_{-\infty}^{\infty}\frac{e^{ax}}{1+e^x}\,dx}_{I}$$

We see that we can again extract the value of the integral that we want to compute.

Let us now compute the contour integral using the Residue theorem. We have 1 pole in the contour, $z = +i\pi$. Looking at the denominator, we have

$$\begin{aligned} 1 + e^z &= 1 + e^{z - i\pi} e^{i\pi} \\ &= 1 - e^{z + i\pi} \\ &= -(z - i\pi) - \frac{1}{2}(z - i\pi)^2 + \dots \end{aligned}$$

Near $z = i\pi$, the integrand is:

$$\frac{e^{az}}{1 + e^z} = \frac{e^{a(i\pi)}}{-(z - i\pi)}$$

Which gives residue $-e^{i\pi a}$. By the Residue Theorem (1.8):

$$\begin{aligned} \oint \frac{e^{az}}{1 + e^z}\, dz &= 2\pi i e^{i\pi a} \\ (1 - e^{2\pi i a}) \underbrace{\int_{-\infty}^{\infty} \frac{e^{ax}}{1 + e^x}\, dx}_{I} &= 2\pi i \left(e^{i\pi a}\right) \\ I &= \frac{\pi}{\sin(\pi a)} \end{aligned}$$

Now let us return to the sides of the contour. Consider the absolute value of the integrand along the right side:

$$\begin{aligned} \left| \frac{e^{az}}{1 + e^z} \right|_{z = R + i\theta} &= \sqrt{\frac{e^{az} e^{a\bar{z}}}{(1 + e^z)(1 + e^{\bar{z}})}} \Big|_{z = R + i\theta} \\ &= \sqrt{\frac{e^{2aR}}{1 + e^R (2\cos\theta) + e^{2R}}} \end{aligned}$$

In the limit of $R \to \infty$, this is dominated by $\frac{1}{e^{2R(1-a)}}$, which vanishes.

Looking at the absolute value of the left side integral:

$$\begin{aligned} \left| \frac{e^{az}}{1 + e^z} \right|_{z = -R + i\theta} &= \sqrt{\frac{e^{az} e^{a\bar{z}}}{(1 + e^z)(1 + e^{\bar{z}})}} \Big|_{z = -R + i\theta} \\ &= \sqrt{\frac{e^{-2aR}}{1 + e^{-R} (2\cos\theta) + e^{-2R}}} \end{aligned}$$

Once again taking the $R \to \infty$ limit, we have $\sqrt{e^{-2aR}}$, which vanishes. Thus, the integral along both side contours vanishes, and our result is correct.

Note that the solution to this problem is very similar to the solution to the previous integral. What we can show is that there is a change of variables that can be used to map the two integrals into each other. If we set $e^x = r$, then $e^x\, dx = dr$, and so we can insert this into the integral:

$$\int_{-\infty}^{\infty} \frac{e^{ax}}{1 + e^x}\, dx = \int_{-\infty}^{\infty} \frac{(e^x)^a}{1 + e^x}\, dx$$

$$= \int_0^\infty \frac{r^a}{1+r}\frac{dr}{r}$$
$$= \int_0^\infty \frac{r^{a-1}}{1+r}\,dr$$

We see that through a simple substitution, we can map the two problems into each other.

Consider the integral

$$I = \int_{-\infty}^{\infty} \frac{\sin x}{x}\,dx$$

Consider the region on the real line bounded by $x = -\epsilon$ and $x = \epsilon$. We will show that as we take the limit $\epsilon \to 0$, the contribution by the bounded region will be -1. Let us consider the integral of just the bounded region:

$$\int_{-\epsilon}^{\epsilon} \frac{\sin x}{x}\,dx = \int_{-\epsilon}^{\epsilon} \left[1 - \frac{x^2}{3!} + \frac{x^4}{5!} + \dots\right] dx$$

Where we have expanded the sin as a Taylor series. We note that this integral is equal to $2\epsilon + \mathcal{O}\left(\epsilon^2\right)$. We see that the contribution from points near $x = 0$ vanishes as $\epsilon \to 0$. We can then consider the original integral, and cut out the region that we know vanishes:

$$\int_{-\infty}^{\infty} \frac{\sin x}{x}\,dx = \lim_{\epsilon\to 0}\left[\int_{-\infty}^{\epsilon} \frac{\sin x}{x}\,dx + \int_{\epsilon}^{\infty} \frac{\sin x}{x}\,dx\right]$$

Consider the following contour integral:

$$\oint_C \frac{e^{iz}}{z}\,dz$$

Where the contour is a half-annulus centered around the origin with outer radius R and inner radius ϵ. This is essentially a semicircle with the ϵ bounded region cut out. We see that the imaginary contribution to this integral along the real line is exactly the remaining portion of the integral that we wish to solve. Note that, by the Residue Theorem (1.8), there are no poles inside the contour and therefore the total integral is 0. We will show that the outer circle contribution vanishes, and thus the integral over the real line and the integral over the inner semicircle must sum to zero. We will compute the inner semicircle integral, and then we will immediately know the contribution of the section along the real line.

To show that the outer circle contribution vanishes, write $z = Re^{i\theta}$, and note that for an upper semicircle, $\theta \in (0, \pi]$. We can write out the function:

$$\frac{e^{iz}}{z} = \frac{e^{iR\cos\theta}e^{-R\sin\theta}}{Re^{i\theta}}$$

And we note that $e^{-R\sin\theta} \to 0$ as $R \to \infty$, and thus the contribution vanishes.

Now we have to compute the inner circle contribution. On this semicircle, we have that $z = \epsilon e^{i\theta}$, where θ starts at π, and goes to 0.

$$\int \frac{e^{iz}}{z}\,dz = \int_{\pi}^{0} \frac{e^{\epsilon e^{i\theta}}}{\epsilon e^{i\theta}}\left(i\epsilon e^{i\theta}\right)\,d\theta$$

$$= -i\pi$$

Now let us put it all together. We know by the Residue theorem that

$$\oint_C \frac{e^{iz}}{z}\,dz = 0$$

And we know that the outer semicircle has no contribution. We know that the inner semicircle contribution exactly opposite to the real line contribution, and we computed it to be $-i\pi$, thus:

$$\begin{aligned}\int_{-R}^{-\epsilon} \frac{e^{iz}}{z}\,dz + \int_{\epsilon}^{R} \frac{e^{iz}}{z}\,dz &= -\left(-i\pi\right)\\ &= i\pi\end{aligned}$$

Where we take the limits $\epsilon \to 0$ and $R \to \infty$.

Now recall that to obtain the original integral, we care about the imaginary component of this, and thus we have that

$$\int_{-\infty}^{\infty} \frac{\sin x}{x}\,dx = \pi$$

Is there a way of setting this problem up by closing the contour in the lower half plane? This can be done by choosing e^{-iz} instead of e^{iz}.

1.7 Principal Value of an Integral

Consider the scenario in which the path of integration actually passes through a singularity in the integrand. In this case, strictly speaking, the integral does not exist. To give it meaning, one must choose a path that circumvents the singularity. How this path is chosen depends on the physics of the problem. The value of the integral then depends on the choice of path.

Consider an integral with a simple pole on the real axis:

$$\int_{-\infty}^{\infty} \frac{f\left(x\right)}{x - x_0}\,dx$$

with x_0 on the real axis, and $f\left(x\right)$ analytic at $x = x_0$.

One possible deformation of the path is the upper semicircle of radius ϵ, denoted $C_>$. We can then split the integral into 3 integrals, one before the semicircle, the semicircle, and the real axis after the semicircle. We denote the *principal value* of the integral to be the contribution from the segments that are not the semicircle, we throw away the contribution due to the semicircle:

$$\int_{-\infty}^{\infty} \frac{f\left(x\right)}{x - x_0}\,dx := \lim_{\epsilon \to 0}\left[\int_{-\infty}^{x_0 - \epsilon} \frac{f\left(x\right)}{x - x_0} + \int_{x_0 + \epsilon}^{\infty} \frac{f\left(x\right)}{x - x_0}\,dx\right]$$

The semicircle integral can be evaluated in the limit $\epsilon \to 0$:

$$\lim_{\epsilon \to 0} \int_{r_>} \frac{f\left(z\right)}{z - x_0}\,dz = -i\pi f\left(x_0\right)$$

Note that if we chose the semicircle that goes below the real axis, we have the same integral, but with a different sign. We see that the value of the integral depends on the path, which must be chosen on physical considerations.

1.8 Mapping

A continuous real function $y = f(x)$ of a real variable x can be exhibited by plotting a curve in the xy plane. This is called a graph.

In the case of a complex function:

$$w = f(z) = u(x, y) + v(x, y)$$

The situation is more complicated because both w and z are represent geometrically by points in the complex plane. This suggests the use of two separate complex plane for the two variables. There is a complex plane for x and y, denoted the z plane, and there is a complex plane for the corresponding point $w = f(z)$, denoted the w plane. Complex functions map points on the z plane to points on the w plane. The point $w = f(z_0)$ corresponding to the point z_0 is called the *image* of z_0 under the transformation $f(z)$. The correspondence between a point in the z plane and its image is called a **mapping**.

Let us consider some simple mappings. Consider the translation:

$$f(z) = z + z_0$$

In terms of x and y:

$$x \to x + x_0$$
$$y \to y + y_0$$

Every point in the z plane is shifted by some constant amount.

Consider a rotation:

$$f(z) = zz_0$$

Where $|z_0| = 1$. This corresponds to rotation by an angle $\phi = \arg(z_0)$. In polar form:

$$f\left(re^{i\theta}\right) = re^{i\theta+\phi}$$

Now consider the case where $z_0 \in \mathbb{R}$, and $z_0 > 0$. In this case, this transformation is a rescaling.

For general z_0, we have a combination of a rotation and a rescaling.

Now consider the mapping $w = f(z) = z^2$. Writing this out in polar form:

$$z = re^{i\theta} \to w = Re^{i\phi}$$

We find that $R = r^2$, and $\phi = 2\theta$.

Taking $0 \leq \theta < 2\pi$, we see that the upper half of the z plane maps onto the entire w plane. The lower half of the z plane also maps onto the entire w plane. The points $re^{i\theta}$ and $re^{i\theta+\pi}$ map to the same point on the w plane. We denote this by saying that the w plane is *covered twice* by the image of the z plane.

We can imagine this as two copies of the w plane placed on top of each other, so that the upper sheet is the image of the upper half z plane, and the lower sheet is the image of the lower half of the z plane. As we pass from the upper half of the z plane to the lower half, the image point pass

from the upper sheet to the lower sheet. This configuration is called a *Riemann surface*. Mapping on to this Riemann surface is one-to-one.

Consider the mapping of inversion:

$$w = f(z) = \frac{1}{z}$$

Writing $z = re^{i\theta}$ and $w = Re^{i\phi}$, we find that $R = \frac{1}{r}$ and $\phi = -\theta$.

This mapping $w = \frac{1}{z}$ maps every circle or straight line onto a circle or straight line. To see this, note that every circle or straight line in the z plane can be written as

$$A\left(x^2 + y^2\right) + Bx + Cy + D = 0$$

where all coefficients are real. Written in terms of z and $\overline{z}$:

$$A(z\overline{z}) + \frac{B}{2}(z + \overline{z}) + \frac{C}{2i}(z - \overline{z}) + D = 0$$

Now we apply the mapping, $w = \frac{1}{z}$, $\overline{w} = \frac{1}{\overline{z}}$:

$$A + \frac{B}{2}(w + \overline{w}) + -\frac{C}{2i}(w - \overline{w}) + Dw\overline{w} = 0$$

Now let $w = u + iv$, and $\overline{w} = u - iv$. We can rewrite what we have as

$$A + Bu - Cv + D\left(u^2 + v^2\right) = 0$$

Which is of the same form as what we started with, and thus we have either a circle or straight line. What is the condition for us to map to a straight line? If $D = 0$ and $A = 0$, then we have a straight line, so a straight line passing through the origin maps through a straight line through the origin.

Also note that a circle that passes through the origin in the z plane will be mapped to a straight line passing through the origin.

1.9 Method of Steepest Descent

This is a method of approximating complex line integrals[1]. We first illustrate the central idea by considering the integral of a real function. We want to look for an approximation to the gamma function $\Gamma(x + 1)$ when x is large, positive, and real. We begin from the definition:

$$\Gamma(x + 1) = \int_0^\infty t^x e^{-t}\, dt$$

If we plot the integrand as a function of t, we see that we have a peak around $t = x$, and the peak gets sharper as x increases. We approximate that the bulk of the contribution to the integral is given by the value at the peak.

[1]This is covered in Mathews and Walker, as well as Arfken

We can first show that it indeed has a peak at $t = x$. Let us denote the integrand $g(t)$. We can find the maximum of $g(t)$:

$$\frac{d}{dt}\left(t^x e^{-t}\right) = 0$$

We can solve this, and we find that the peak t_0 is found at x. We approximate the integrand as:

$$g(t) = e^{f(t)}$$

Where $f(t) = x \log t - t$. We can now compute derivatives of $f(t)$:

$$\frac{d}{dt} f(t) = \frac{x}{t} - 1$$
$$\frac{d^2}{dt^2} f(t) = -\frac{x}{t^2}$$

Now Taylor expanding $f(t)$ about $t = x$:

$$f(t) = (x \log x - x) - \frac{1}{2}\frac{1}{x}(t-x)^2 + \ldots$$

We can thus approximate the integral as

$$\begin{aligned}\Gamma(x+1) &\approx \int_0^\infty \exp\left[x \log x - x - \frac{1}{2x}(t-x)^2\right] dt \\ &\approx e^{x \log x - x} \int_0^\infty e^{-\frac{1}{2x}(t-x)^2} dt \\ &\approx e^{x \log x - x} \int_{-\infty}^\infty e^{-\frac{1}{2x}(t-x)^2} dt\end{aligned}$$

Where we have increased the bounds to the real numbers by noting that the contribution from $-\infty \to 0$ produces negligible effects on the integral. The remaining integral is a standard Gaussian integral:

$$\begin{aligned}\Gamma(x+1) &\approx e^{x \log x - x}\sqrt{2\pi x} \\ &= \sqrt{2\pi x} x^x e^{-x}\end{aligned}$$

This is the first term in Stirling's formula:

$$n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

Which is valid for large n.

Let us now consider the complex case, which is significantly more complicated. Usually, when we apply the method of steepest descent, we are integrating a function of the form

$$I(s) = \int_C g(z) e^{sf(z)} dz$$

Where s is large and positive, and g and f are analytic functions. In the case where $f(z) = u + iv$, we would expect most of the contribution to the integral to come from the region where u is maximized,

because we have e^{su}. However, the phase e^{iv} could have either sign. We have to choose the contour such that u is very large, but the phase picked up by v does not go negative.

Since $f(z)$ is an analytic function, neither u nor v can have an extremum other than at a singularity:

$$f(z) = u + iv$$

If $f(z)$ is analytic, $\nabla^2 u = \nabla^2 v = 0$, the two functions satisfy Laplace's equation. Recall that Laplace's equation has no maxima or minima in a bounded region. Thus u and v have no maxima in regions in which they are analytic. If there is a point where their derivatives are zero, the point must be a saddle point. At the saddle point, we can approximate $f(z)$ as:

$$f(z) = f(z_0) + \frac{1}{2} f''(z_0)(z - z_0)^2$$

Where $z = z_0$ is the location of the saddle point. We can write then write things as an absolute value times a phase:

$$f''(z_0) = \left|f''(z_0)\right| e^{i\theta}$$
$$z - z_0 = |z - z_0| e^{i\alpha}$$

Where α controls the direction of z from z_0. We can now break $f(z)$ into the real and imaginary parts:

$$u(x,y) \approx u(x_0,y_0) + \frac{1}{2}\left|f''(z_0)\right| |z - z_0|^2 \cos(\theta + 2\alpha)$$
$$v(x,y) \approx v(x_0,y_0) + \frac{1}{2}\left|f''(z_0)\right| |z - z_0|^2 \sin(\theta + 2\alpha)$$

We see that looking at u, the path of steepest descent is when $\cos(\theta + 2\alpha) = -1$, which corresponds to $\alpha = -\frac{\theta}{2} \pm \frac{\pi}{2}$. Along these directions, $\sin(\theta + 2\alpha)$ is a constant, so the oscillatory factor in the integral will not cause any problems.

We choose the contour between two points A and B such that the contour passes over the saddle point, along the path of steepest descent. this will give us a maximum of $v(x,y)$, and minimal cancellations from $v(x,y)$.

We write $z = z_0 + le^{i\alpha}$, where l is real and positive. The integral can be written as

$$I(s) = \int_{C'} g\left(z_0 + le^{i\alpha}\right) \exp\left[sf\left(z_0 + le^{i\alpha}\right)\right] e^{i\alpha}\, dl$$

Now Taylor expanding:

$$f\left(z_0 + le^{i\alpha}\right) = f(z_0) + \frac{1}{2} \underbrace{f''(z_0)}_{|f''(z_0)|e^{i\theta}} l^2 e^{2i\alpha} + \dots$$
$$g\left(z_0 + le^{i\alpha}\right) = g(z_0) + \dots$$

Assuming that $g(z)$ is slowly varying around z_0, we can approximate the integral as

$$I(s) \approx g(z_0)\, e^{sf(z_0)} \int_{C'} e^{-\frac{1}{2}|f''(z_0)|l^2 s} e^{i\alpha}\, dl$$

Since the integrand falls away very quickly away from z_0, we can replace the limits on l with $-\infty$ to ∞. This produces a standard Gaussian integral:

$$I(s) \approx \frac{\sqrt{2\pi} g(z_0) e^{sf(z_0)} e^{i\alpha}}{\sqrt{s\left|f''(z_0)\right|}}$$

Note that we have a dependence on α, which is equal to

$$\alpha = -\frac{\theta}{2} \pm \frac{\pi}{2}$$

where the sign is determined by the direction in which we traverse the saddle point, and θ is the phase of the second derivative $f''(z_0)$.

Let us consider some examples.

The gamma function can be analytically continued for complex values:

$$\Gamma(z+1) = \int_0^\alpha e^{-t} t^z \, dt \qquad \mathrm{Re}(z) > -1$$

where the integral is along the real line. If we write $z = se^{i\beta}$, we want an approximation for $\Gamma(z)$ when $s \gg 1$.

Let us look at the general form that we have derived an approximation for:

$$I(s) = \int g(z) e^{sf(z)} \, dz$$

Naively, we can make the mapping $g(z) = e^{-z}$, and $f(z) = e^{i\beta} \ln z$. However, we want a place where $f'(z) = 0$, and our choice of $f(z)$ has no saddle point. Thus we have to make a different choice.

To do so, we rewrite the integral:

$$\begin{aligned}\Gamma(z+1) &= \int_0^\infty e^{z \ln t - t} \, dt \\ &= \int_0^\infty e^{s\left[e^{i\beta} \ln t - \frac{t}{s}\right]} \, dt\end{aligned}$$

This is of the form $\int_0^\infty e^{sf(t)} \, dt$, with $f(t) = e^{-\beta} \ln t - \frac{t}{s}$. We see that with this choice, $g(z) = 1$.

We can compute the derivatives of $f(t)$:

$$\begin{aligned}f'(t) &= \frac{e^{i\beta}}{t} - \frac{1}{s} \\ f''(t) &= -\frac{e^{i\beta}}{t^2}\end{aligned}$$

Setting $f'(t) = 0$, we find that $t_0 = se^{i\beta} = z$, and thus at $t = t_0$:

$$f(t_0) = e^{i\beta} \left[\ln z - 1\right]$$

And we can compute the second derivative:

$$\begin{aligned} f''(t_0) &= -\frac{e^{-i\beta}}{s^2} \\ &= \frac{1}{s^2} e^{i(\pi-\beta)} \end{aligned}$$

Thus we see that $|f''(t_0)| = \frac{1}{s^2}$, and $\theta = \pi - \beta$. We can then compute α:

$$\alpha = -\left(\frac{\pi - \beta}{2}\right) \pm \frac{\pi}{2}$$

Which gives us $\alpha = \frac{\beta}{2}$ or $\alpha = \frac{\beta}{2} - \pi$. Now applying the general formula:

$$I(s) = \frac{\sqrt{2\pi} g(t_0) e^{sf(t_0)} e^{i\alpha}}{\sqrt{s |f''(t_0)|}}$$

From this, we have that

$$\begin{aligned} \Gamma(z+1) &\approx \sqrt{2\pi s} e^{s\left[e^{i\beta}(\ln z - 1)\right]} e^{i\alpha} \\ &\approx \sqrt{2\pi s} e^{[z \ln z - z]} e^{i\alpha} \end{aligned}$$

Now we have to choose which α to use, which determines the sign. Consider the case where $\beta = 0$. We expect to get a positive approximation for $\Gamma(z+1)$, and thus we must pick the α that produces a positive result. We see that $e^{i0} = +1$, while $e^{-i\pi} = -1$. Thus we must choose $\alpha = \frac{\beta}{2}$:

$$\begin{aligned} \Gamma(z+1) &\approx \sqrt{2\pi s} e^{[z \ln z - z]} e^{\frac{i\beta}{2}} \\ &= \sqrt{2\pi} z^{\left(z+\frac{1}{2}\right)} e^{-z} \end{aligned}$$

Let us do an example. Consider the Hankel function of the first kind, which can be represented by the contour integral:

$$H_\nu^{(1)}(s) = \frac{1}{\pi} \int_\epsilon^{\infty e^{i\pi}} e^{\frac{s}{2}\left(z - \frac{1}{z}\right)} \frac{1}{z^{\nu+1}} \, dz$$

where $\epsilon \to 0$. Since this function has a branch cut on the entire negative real line, the integral bounds represent a line that is just above 0, and going all the way to negative infinity.

This integral is in the standard form for our steepest descent approximation:

$$\begin{aligned} g(z) &= \frac{1}{z^{\nu+1}} \\ f(z) &= \frac{1}{2}\left(z - \frac{1}{z}\right) \end{aligned}$$

Taking derivatives:

$$\begin{aligned} f'(z) &= \frac{1}{2} - \frac{1}{2z^2} \\ f''(z) &= -\frac{1}{z^3} \end{aligned}$$

We see that z_0 is i or $-i$. We use the contour that passes through the upper point, $+i$, and then goes to $-\infty$ from above the real axis.

At $z = +i$, $f(z_0) = i = e^{i\frac{\pi}{2}}$, and $f''(z) = -i = e^{-i\frac{\pi}{2}}$. Note that the argument of z must be between $-\pi$ and π in order to avoid the branch cut. We see that $\theta = -\frac{\pi}{2}$, and thus

$$\alpha = -\frac{\pi}{2} \pm \frac{\pi}{2}$$

And so $\alpha = \frac{3\pi}{4}$ or $-\frac{\pi}{4}$. By inspecting the contour, we see that $\alpha = \frac{3\pi}{4}$ must be the correct choice.

We can then write out the result of the approximation:

$$H_\nu^{(1)}(s) \approx \sqrt{\frac{2}{\pi s}} e^{i\left(s - \nu\frac{\pi}{2} + \frac{\pi}{4}\right)}$$

1.10 Analytic Continuation

Earlier, we discussed Cauchy's Integral Formula (1.5), which told us that knowing the value of the function on every point of a closed contour can give us the value inside the region. Is there a more powerful statement? Suppose we have a large region, on which $f(z)$ is analytic. If we know the value of $f(z)$ for some subcontour of this region, then we can actually determine $f(z)$ everywhere in the large region. In fact, it does not need to be a closed contour, if we know the value on a line in the region, we can obtain the value of $f(z)$ for the rest of the region.

We begin with the identity theorem, which claims that if two analytic functions are equal in a subregion of a region, then they are equal across the entire region.

Theorem 1.9. ***Identity Theorem.*** *Let $f_1(z)$ and $f_2(z)$ be two functions of z that are analytic in a region D. If the two functions coincide in the neighborhood of a point z_0 in D or on the segment of a curve lying in D, then they coincide throughout D.*

Proof. The proof of the identity theorem is in two steps. First, we show that if a function $f(z)$ is analytic, the points where $f(z) = 0$ are isolated unless $f(z) = 0$ in the entire region.

From this, we consider the function $f(z) = f_1(z) - f_2(z)$. Since the two functions are analytic, then $f(z)$ is analytic. Then we note that the zeros of this function (where $f_1(z) = f_2(z)$) are not isolated, and thus by the previous statement, $f(z) = 0$, and thus $f_1(z) = f_2(z)$ on the region.

Let us now prove our first statement. If a function $f(z) = 0$ at a point $z = z_0$, this point is called a *zero* of $f(z)$. A function is said to have a zero of order n at $z = z_0$ if

$$f(z_0) = 0, \qquad \frac{df}{dz}\Big|_{z=z_0} = 0, \ldots, \frac{d^{n-1}}{dz^{n-1}} f(z)\Big|_{z=z_0} = 0$$

but $\frac{d^n}{dz^n} f(z)\big|_{z=z_0} \neq 0$. That is, the nth derivative is nonzero, and all lower order derivatives are zero.

If $f(z)$ has an nth order zero, the Taylor expansion of $f(z)$ around z_0 is given by

$$\begin{aligned} f(z) &= a_n (z - z_0)^n + a_{n+1} (z - z_0)^{n+1} \ldots \\ &= (z - z_0)^n \sum_{k=0}^{\infty} a_{n+k} (z - z_0)^k \end{aligned}$$

$$= (z - z_0)^n h(z)$$

Where $h(z) = \sum_{k=0}^{\infty} a_{n+k} (z - z_0)^k$. Since $h(z)$ is analytic and nonvanishing at $z = z_0$, and since $h(z)$ is continuous, it is nonvanishing in some neighborhood of z_0. Then $f(z)$ is nonvanishing in the neighborhood of z_0. This implies that the zeros of $f(z)$ are isolated. The only case where this is not true is when all the derivatives of $f(z)$ are zero, in which case the function is 0 everywhere. Thus we have proven the first statement, and the second statement follows.

□

Consider a function that is analytic everywhere except for a branch cut. Suppose we only know the value of the function in the neighborhood of a point z_0. The claim is that, from the value of $f(z)$ in the region around z_0, we have enough information to determine $f(z)$ everywhere where the function is analytic.

Suppose we want to find the value of $f(z)$ at $z = z_0'$. To do this, we draw a smooth line connecting z_0 to z_0', making sure to never cross into a region in which $f(z)$ is not analytic. We know the Taylor expansion of $f(z)$ around $z = z_0$:

$$f(z) = \sum_{n=0}^{\infty} a_n^{(0)} (z - z_0)^n$$

By Laurent's theorem (1.6), this series is valid in the largest circle that can be drawn that does not pass a singular point. However, if this largest possible circle does not contain z_0', we must continue. We pick another point $z_1$on the curve we drew, and then do a Taylor series around that point, since we now know the Taylor series around z_1. We can then draw the largest circle for which this Taylor series is valid, and then we can repeat this process of picking new points and finding circles for which the Taylor expansions are valid, until we draw a circle that contains z_0'.

Let our largest circle be known as γ_0, we choose a point z_1 inside γ_0, and since $f(z)$ is known inside γ_0, we can determine the coefficients of the Taylor expansion of $f(z)$ around $z = z_1$, by computing all of the derivatives:

$$f(z) = \sum_{n=0}^{\infty} a_n^{(1)} (z - z_1)^n$$

This Taylor expansion is valid in the largest possible circle around z_1 that does not hit a singular point, which we denote γ_1. We have now found the value of $f(z)$ at some point that is in γ_1 but not in γ_0. We then choose z_2 in γ_1 but not in γ_0, and repeat the process of computing the Taylor expansion and drawing the largest circle. We will eventually find a circle that contains z_0', providing the value of $f(z)$ at $z = z_0'$.

The process of determining the behaviour of an analytic function outside the region where it was originally defined is called *analytic continuation*.

Consider the following situation. We have a region D_1, and a partially overlapping region D_2. If $f_1(z)$ is analytic in D_1, $f_2(z)$ is analytic in D_2, and $f_1(z) = f_2(z)$ in the region where D_1 and D_2 overlap, the analytic continuation of f_1 into D_2 must be equal to D_2.

For example, consider the function $f_1(z) = 1 + z + z^2 + \dots$ for $|z| < 1$, and the function $f_2(z) = \frac{2}{3} + \left(\frac{2}{3}\right)^2 \left(z + \frac{1}{2}\right) + \left(\frac{2}{3}\right)^3 \left(z + \frac{1}{2}\right)^2 + \dots$, valid for $\left|z + \frac{1}{2}\right| < 1$ z.

These are geometric series, so they can be summed:

$$\begin{aligned} f_1(z) &= \frac{1}{1-z} \qquad \text{for } |z|<1 \\ f_2(z) &= \frac{\frac{2}{3}}{1-\frac{2}{3}\left(z+\frac{1}{2}\right)} \\ &= \frac{1}{1-z} \qquad \text{for } \left|z+\frac{1}{2}\right| < \frac{3}{2} \end{aligned}$$

The circle $|z|<1$ is surrounded by the circle given by $\left|z+\frac{1}{2}\right| < \frac{3}{2}$.

We see that if we analytically continue $f_1(z)$ into the region for which $f_2(z)$ is defined, we get the same result as if we used $f_2(z)$.

Consider the function $g(x) = \sin x$, for $-\pi < x < \pi$. There is a unique analytical continuation of this function into the complex plane. The continuation is just $\sin z$. This is equal to $\sin x$ on the real line, and it is analytic on the real line, and therefore is the unique analytic continuation.

Now consider

$$f(x) = 1 + x + x^2 + \ldots \qquad 0 << \frac{1}{2}$$

The analytic continuation of this is obtained by once again replacing x with z:

$$\begin{aligned} f(z) &= 1 + z + z^2 + \ldots \qquad 0 < |z| < 1 \\ &= \frac{1}{1-z} \end{aligned}$$

However, when we sum this geometric series, we see that we only have a singularity at $z=1$. Other than $z=1$, the function is analytic everywhere. Thus, we can analytically continue the function to a larger region than just the circle with radius 1.

Let us do an example. Consider the integral:

$$I\left(a^2\right) = \int_{-\infty}^{\infty} \frac{dx}{x^2 - a^2 + i\epsilon} \qquad \epsilon \to 0^+$$

Suppose we first want to evaluate this integral for $a^2 \in \mathbb{R}^+$, and then for $a^2 \in \mathbb{R}^-$. By promoting a^2 to a complex variable, and analytically continuing $I\left(a^2\right)$ appropriately, we can show that results for both cases are consistent.

We can first compute the real and positive case. The integrand is singular at $z = -a + i\hat{\varepsilon}$ and $z = a - i\hat{\varepsilon}$, where $\hat{\varepsilon} = \frac{\varepsilon}{2a}$. If we examine the pole structure, we have a pole slightly above the negative real axis, and a pole slightly below the positive real axis, corresponding to $-a + i\hat{\varepsilon}$ and $z = a - i\hat{\varepsilon}$, respectively. We close the contour in the upper half plane, so only the pole at $z = -a + i\hat{\varepsilon}$ will contribute. From this contour, we will find that

$$I\left(a^2\right) = -\frac{i\pi}{a}$$

For the case where a^2 is negative and real, we have a standard integral when $\epsilon \to 0^+$:

$$I = \int_{-\infty}^{\infty} \frac{dz}{z^2 + |a^2| + i\varepsilon}$$

$$= \frac{1}{|a|} \arctan\left(\frac{z}{|a|}\right)\Big|_{-\infty}^{\infty}$$
$$= \frac{\pi}{|a|}$$

Now we want to see how these two are consistent with each other. If a^2 is treated as a complex variable, we want to examine the singularity structure of $I\left(a^2\right)$. We note that if $\text{Im}\left(a^2\right) \leq 0$, then $I\left(a^2\right)$ is analytic. We can then write a^2 as:

$$a^2 = |a|^2 e^{i\theta} \qquad -\pi < \theta < 0$$

From the first case, we found that $I\left(a^2\right) = -\frac{\pi i}{a}$ for $\theta = 0$. We can then analytically continue this into the region for $-\theta$ between 0 and $-\pi$:

$$I\left(a^2\right) = -\frac{\pi i}{a}$$
$$= -\frac{\pi i}{|a|e^{i\theta/2}}$$

Setting $\theta = -\pi$, which corresponds to $a^2 = -|a|^2$, we find that

$$I\left(a^2\right) = -\frac{\pi i}{|a|e^{i\theta/2}}$$
$$= \frac{\pi}{|a|}$$

Which agrees with what we computed manually.

2 Classical Mechanics

2.1 Calculus of Variations

Calculus of variations is the calculation of extrema of quantities which can be expressed as an integral.

For example, given two points on a plane, what is the shortest distance between them? Obviously, the answer is a line connecting the two points, but how do we go about proving this?

First, let us consider the segment length of the path $y(x)$ joining (x_1, y_1) to (x_2, y_2). The segment length is given by

$$\begin{aligned} ds &= \sqrt{dx^2 + dy^2} \\ &= dx\sqrt{1 + \left(\frac{dy}{dx}\right)^2} \end{aligned}$$

We can define the length of the path as the integral over all path segments:

$$\begin{aligned} L &= \int_1^2 ds \\ &= \int_{x_1}^{x_2} \sqrt{1 + y'^2}\, dx \end{aligned}$$

We now want to minimize L with respect to $y(x)$. Note that L is a *functional*, a function of a function, $L[y(x)]$.

Let us define $y_0(x)$ as the function that minimizes L. Now consider some more general $y(x)$, which is a small perturbation from $y_0(x)$:

$$y = y_0(x) + \varepsilon\eta(x)$$

Where $\varepsilon \ll 1$, and $\eta(x)$ is some general function. Since $y_0(x)$ is the minimal path, then $L[y_0(x)] < L(y(x))$, for any value of η and ε. Let us consider the length for the new function:

$$\begin{aligned} L[y_0(x) + \varepsilon\eta(x)] &= \int \sqrt{1 + (y_0' + \varepsilon\eta')^2}\, dx \\ &= \int dx\, \sqrt{1 + y_0'^2 + 2\varepsilon\eta' y_0'} + \mathcal{O}(\varepsilon^2) \\ &= \int dx\, \sqrt{1 + y_0'^2} + \int dx\, \frac{\varepsilon y_0'\eta'}{\sqrt{1 + y_0'^2}} + \mathcal{O}\left(\varepsilon^2\right) \end{aligned}$$

Where we have expanded for small ε and then used a binomial expansion. Rewriting the second term as a total derivative:

$$\begin{aligned} L[y_0 + \varepsilon\eta] &= \int dx\, \sqrt{1 + y_0'^2} + \int dx\, \frac{\varepsilon y_0'\eta'}{\sqrt{1 + y_0'^2}} + \mathcal{O}\left(\varepsilon^2\right) \\ &= L[y_0(x)] - \int_{x_1}^{x_2} dx\, \varepsilon \frac{d}{dx}\left[\frac{y_0'}{\sqrt{1 + y_0'^2}}\right]\eta \end{aligned}$$

Now we note that in order for the relationship between the length of y and y_0 to be maintained, we need the second term to go to zero, since ε can take any sign, and therefore this term could push the path length below the minimum path length of y_0. Thus we have that

$$\int_{x_1}^{x_2} \frac{d}{dx}\left[\frac{\varepsilon y_0'}{\sqrt{1+y_0'^2}}\right]\eta\, dx = 0$$

For all $\eta(x)$. We can see that for this to be true for all η, we need the derivative term to be 0:

$$\frac{d}{dx}\left[\frac{\varepsilon y_0'}{\sqrt{1+y_0'^2}}\right] = 0$$

This implies that

$$\frac{y_0'}{\sqrt{1+y_0'^2}} = \text{const.}$$

Which then tells us that y_0' must be a constant, and thus $y_0(x)$ is a line.

Let us do a similar process to derive the Euler-Lagrange equation.

2.2 Euler-Lagrange Equation

Consider some function S, which is dependent on some functional f :

$$S = \int_{x_1}^{x_2} f\left[y(x), y'(x), x\right]\, dx$$

Where $y(x_1) = y_1$, and $y(x_2) = y_2$. Essentially, we are connecting two points in the plane. We want to find the function f that extremizes S.

Once again, let $y = y_0(x)$ extremize S. Now consider $y = y_0(x) + \varepsilon\eta(x)$. We have the constraint that

$$\left.\frac{dS}{d\varepsilon}\right|_{\varepsilon=0} = 0$$

in order to maintain the fact that y_0 extremizes S. Let us now write S as a function of ε:

$$S(\varepsilon) = \int dx\, f\left(y_0 + \varepsilon\eta, y_0' + \varepsilon\eta', x\right)$$

Expanding for small ε:

$$f\left(y_0 + \varepsilon\eta, y_0' + \varepsilon\eta', x\right) = f\left(y_0, y_0', x\right) + \varepsilon\eta\left(\frac{df}{dy}\right)_{y_0', x} + \varepsilon\eta'\left(\frac{df}{dy'}\right)_{y_0, x}$$

Now writing out the derivative $\frac{dS}{d\varepsilon}$:

$$\frac{dS}{d\varepsilon} = \int dx\, \left[\eta\left(\frac{df}{dy}\right) + \eta'\left(\frac{df}{dy'}\right)\right]$$

Now integrating the second term by parts (and noting that the extra term goes to zero because η has to be zero at the boundary):

$$\begin{aligned}\frac{dS}{d\varepsilon} &= \int dx \left[\eta\left(\frac{df}{dy}\right) - \eta\frac{d}{dx}\left(\frac{df}{dy}\right)\right] \\ &= \int dx\, \eta\left[\frac{df}{dy} - \frac{d}{dx}\frac{df}{dy}\right]\end{aligned}$$

Now noting that this must, by the constraints, be zero:

$$\int dx\, \eta\left[\frac{df}{dy} - \frac{d}{dx}\frac{df}{dy}\right] = 0$$

And once again, for this to be true for general η, we need the other part to be zero:

$$\boxed{\frac{df}{dy} - \frac{d}{dx}\frac{df}{dy} = 0} \tag{4}$$

Which is the Euler-Lagrange equation.

Let us do an example, returning to our question about the shortest path between two points. In this case, we had a length function:

$$L = \int_{x_1}^{x_2} \sqrt{1 + y'^2}\, dx$$

We see that $L = f\left(y, y', x\right) = \sqrt{1 + y'^2}$, which we can insert into the Euler-Lagrange equation:

$$0 - \frac{d}{dx}\left(\frac{y'}{\sqrt{1 + y'^2}}\right) = 0$$

From which we can once again conclude that y' is a constant, and therefore y must be a line.

2.3 The Brachistochrone

Given two points 1 and 2, in what shape should a track be built so that a particle released from point 1 will reach point 2 in the smallest amount of time? This is the Brachistochrone (equal time) problem, and is a classic example of the application of the Euler-Lagrange equation. We want to minimize the time:

$$\begin{aligned}T &= \int_1^2 \frac{ds}{v} \\ &= \int_1^2 dx\, \frac{\sqrt{1 + y'^2}}{\sqrt{2gy}}\end{aligned}$$

Where we have derived the velocity from energy conservation:

$$\begin{aligned}\frac{1}{2}mv^2 &= mgy \\ v &= \sqrt{2gy}\end{aligned}$$

Thus we have that our f that we will insert into Euler-Lagrange:

$$f = \frac{\sqrt{1+y'^2}}{\sqrt{2gy}}$$

Computing the derivatives:

$$\frac{df}{dy'} = \frac{y'}{\sqrt{2gy\left(1+y'^2\right)}}$$
$$\frac{df}{dy} = \frac{1}{2}\frac{\sqrt{1+y'^2}}{\sqrt{2gy^3}}$$

Plugging this into the Euler-Lagrange equation:

$$-\frac{1}{2}\frac{\sqrt{1+y'^2}}{\sqrt{2gy'^3}} - \frac{d}{dx}\left[\frac{y'}{\sqrt{2gy\left(1+y'^2\right)}}\right] = 0$$
$$\frac{d}{dx}\left[\frac{y'}{\sqrt{y\left(1+y'^2\right)}}\right] + \frac{\sqrt{1+y'^2}}{2\sqrt{y^3}} = 0$$

Multiplying by $y'/\sqrt{y\left(1+y'^2\right)}$:

$$\frac{y'}{\sqrt{1+y'^2}}\frac{d}{dx}\left[\frac{y'}{\sqrt{1+y'^2}}\right] + \frac{y'}{2y^2} = 0$$
$$\frac{d}{dx}\left[\frac{1}{2}\frac{y'^2}{1+y'^2} - \frac{1}{2y}\right] = 0$$
$$\frac{y'^2}{1+y'^2} - 1 = \text{const} \cdot y$$

Since we are considering y as a height, let us define y to be positive, and we then note that the left side of the expression is negative. Thus, the constant on the right side must be negative, and by dimensional analysis, it have units of one over length. For this reason, we denote the constant to be $-\frac{1}{2a}$, where a is a positive length constant:

$$\frac{y'^2}{1+y'^2} - 1 = -\frac{1}{2a}y$$

Solving this equation for y':

$$y' = \frac{\sqrt{2a-y}}{y}$$

Now trying to solve for $y\left(x\right)$:

$$\int dx = \int dy\sqrt{\frac{y}{2a-y}}$$

We now apply the ansatz $y = a\left(1-\cos\theta\right)$:

$$\int dx = \int d\theta\ \left(a\sin\theta\right)\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}$$

Now we write the $\sin\theta$ in terms of the half angle formula, and write the square root as a tangent half angle formula:

$$\int dx = \int d\theta\, 2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)\tan\frac{\theta}{2}$$
$$\int dx = 2a\int d\theta\, \sin^2\frac{\theta}{2}$$
$$\int dx = a\int d\theta\, (1-\cos\theta)$$

From this, we find that $x = a\,(\theta - \sin\theta)$.

Thus, we have that $x = a\,(\theta - \sin\theta)$, and $y = a\,(1-\cos\theta)$.

If we consider this parametric function close to point 1, where $x \ll a$ and $y \ll a$, we can expand:

$$x = \frac{a\theta^3}{6} \qquad y = \frac{a\theta^2}{2}$$
$$\theta = \left(\frac{6x}{a}\right)^{1/3}$$

From this, we find that close to point 1:

$$y = \left(\frac{9ax^2}{2}\right)^{1/3}$$

We see that close to the first point, we have a very steep descent.

Consider a functional $I\,(q_1, q_2, \ldots, \dot{q_1}, \dot{q_2}, \ldots, t)$. From this, we can use a similar process as the single-variable case, we can derive a set of Euler-Lagrange equations:

$$\frac{\partial I}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial I}{\partial \dot{q_i}}\right) = 0$$

2.4 Lagrangian Mechanics

The Lagrangian formulation of mechanics is an alternative to the Newtonian mechanics, and provides two advantages. The first is that it works equally well in all coordinate systems. The second is that it handles constrained systems easily, and unless desired, we can bypass the computation of the forces of constraint.

Let us begin with unconstrained motion. Consider a single particle in 3 dimensions, moving unconstrained under the action of a force $\boldsymbol{F} = -\nabla U$. The kinetic energy is given by $T = \frac{1}{2}m\dot{\boldsymbol{x}}^2 = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right)$. By definition, the Lagrangian is given by the difference between the kinetic and potential energy:

$$\mathcal{L} = T - U$$

Computing the derivative with respect to a coordinate and its time derivative:

$$\left(\frac{\partial \mathcal{L}}{\partial x}\right)_{\dot{x}} = -\frac{\partial U}{\partial x}$$

$$\begin{aligned} &= F_x \\ \left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right)_x &= m\dot{x} \\ &= p_x \end{aligned}$$

From Newton's second law, $\frac{d}{dt}p_x = F_x$, and thus we have that

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right)_x = \frac{\partial \mathcal{L}}{\partial x}_{\dot{x}}$$

Writing the same equation for any coordinate:

$$-\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}_i}\right) + \frac{\partial \mathcal{L}}{\partial x_i} = 0$$

Now, we make a crucial observation, this has the form of the Euler-Lagrange equation (4), obtained by extremizing $S = \int_{t_1}^{t_2} \mathcal{L}\, dt$.

The action S will be stationary along the physical path between the endpoints (t_1, x_1, y_1, z_1) and (t_2, x_2, y_2, z_2). This is known as *Hamilton's Principle*.

One advantage of the Lagrangian formalism is that it works for generalized coordinates, not just Cartesian. If we switch from (x, y, z) ot some generalized (q_1, q_2, q_3), as long as there is a one to one mapping between the two coordinate systems, we can work in that coordinate system. This means that we can choose the coordinate system that best suits the problem, rather than just working in Cartesian.

We can write our Lagrangian as a function of our generalized coordinates:

$$\mathcal{L}\left(x, y, z, \dot{x}, \dot{y}, \dot{z}\right) \to \mathcal{L}\left(q_i, \dot{q}_i\right)$$

The action integral can then be written as

$$S = \int_{t_1}^{t_2} \mathcal{L}\left(q_i, \dot{q}_i\right)\, dt$$

Consider two points in the Cartesian plane, (t_1, x_1, y_1, z_1) and (t_2, x_2, y_2, z_2), and some path connecting the two of the points. At every point along the trajectory, the value of $\mathcal{L}$ is fixed. If we switch coordinate systems, the value of the Lagrangian is invariant, since we define the Lagrangian to be the same regardless of coordinate system. Thus, the value of the action for a given path must be the same in either coordinate system, since we are integrating over the Lagrangian, which is coordinate invariant. Thus the path that extremizes S will be the same across both coordinate systems. For this reason, we can apply the Euler-Lagrange equations for generalized coordinates:

$$-\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) + \frac{\partial \mathcal{L}}{\partial q_i} = 0$$

We denote $\frac{\partial \mathcal{L}}{\partial q_i}$ as the ith component of the generalized force, and denote $\frac{\partial \mathcal{L}}{\partial \dot{q}_i}$ as the ith component of the generalized momentum. From this, we see that the Euler-Lagrange equation states that the rate of change of generalized momentum is equal to the generalized force.

Let us do an example. Consider a single particle in polar coordinates. This has kinetic energy given by

$$\begin{aligned} T &= \frac{1}{2}mv^2 \\ &= \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\phi}^2\right) \end{aligned}$$

Which can be derived by noting that $\boldsymbol{r} = r\hat{e}_r$, and the velocity is given by

$$\begin{aligned} \boldsymbol{v} &= \frac{d\boldsymbol{r}}{dt} \\ &= \dot{r}\hat{e}_r + r\frac{d\hat{e}_r}{dt} \end{aligned}$$

This derivative term is nonzero. If we change r, the direction of $\hat{e}_r$ does not change, so $\hat{e}_r$ must only depend on ϕ. Thus, we can write the time derivative of the direction as:

$$\frac{d\hat{e}_r}{dt} = \frac{d\hat{e}_r}{d\phi}\frac{d\phi}{dt}$$

Now we note that $\frac{d\hat{e}_r}{d\phi} = \hat{e}_\phi$, and thus we have that

$$\boldsymbol{v} = \dot{r}\hat{e}_r + r\dot{\phi}\hat{e}_\phi$$

Squaring this:

$$\boldsymbol{v}^2 = \dot{r}^2 + r^2\dot{\phi}^2$$

We can now write out the Lagrangian:

$$\begin{aligned} \mathcal{L} &= T - U \\ &= \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\phi}^2\right) - U\left(r, \phi\right) \end{aligned}$$

We have two Euler-Lagrange equations:

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{r}}\right) &= \frac{\partial\mathcal{L}}{\partial r} \\ \frac{d}{dt}\left(m\dot{r}\right) &= mr\dot{\phi}^2 - \frac{\partial U}{\partial r} \\ m\ddot{r} - mr\dot{\phi}^2 &= -\frac{\partial U}{\partial r} \end{aligned}$$

$\ddot{r}$ is the radial acceleration, and $r\dot{\phi}^2$ is the centripetal acceleration. Together, the two of them balance the radial force, $-\frac{\partial U}{\partial r}$.

Looking at the second Euler-Lagrange equation:

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\phi}}\right) &= \frac{\partial\mathcal{L}}{\partial\phi} \\ \frac{d}{dt}\left(mr^2\dot{\phi}\right) &= -\frac{\partial U}{\partial\phi} \end{aligned}$$

We see that $mr^2\dot{\phi}$ is the angular momentum:

$$\begin{aligned}\boldsymbol{L} &= \boldsymbol{r} \times \boldsymbol{p} \\ &= \boldsymbol{r} \times m\boldsymbol{v} \\ &= m\left(\boldsymbol{r} \times \left[\dot{r}\hat{e}_r + r\dot{\phi}\hat{e}_\phi\right]\right) \\ &= mr^2\dot{\phi}\hat{e}_z\end{aligned}$$

If the left side is the rate of change in the angular momentum, then the right side must be the torque, which we can verify:

$$\begin{aligned}\boldsymbol{F} &= -\nabla U \\ &= -\frac{\partial U}{\partial r}\hat{e}_r - \frac{1}{r}\frac{\partial U}{\partial \phi}\hat{e}_\phi\end{aligned}$$

The torque is defined as:

$$\begin{aligned}\boldsymbol{\tau} &= \boldsymbol{r} \times \boldsymbol{F} \\ &= r\hat{e}_r \times \left(-\frac{\partial U}{\partial r}\hat{e}_r - \frac{1}{r}\frac{\partial U}{\partial \phi}\hat{e}_\phi\right) \\ &= -\frac{\partial U}{\partial \phi}\hat{e}_z\end{aligned}$$

Note that in this case, the generalized momentum was the angular momentum, and the generalized force was the torque, which are both dimensionally different from the usual dimensions of momentum and force.

2.4.1 Constrained Systems

Now let consider constrained systems.

Consider a simple pendulum. In this system, x and y are not independent, the length of the string dictates that $\sqrt{x^2 + y^2} = l$. The system has only 1 degree of freedom, rather than 2. The most convenient parameterization of this system to us the angle from the vertical, θ. If we know θ, we can find x and y. A tremendous advantage of the Lagrangian formalism is that the Euler-Lagrange equations apply directly, we do not need to factor in the constraints. We will show this to be true later, but for now, let us assume it to be true.

We can write the kinetic energy:

$$\begin{aligned}T &= \frac{1}{2}mv^2 \\ &= \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) \\ &= \frac{1}{2}ml^2\dot{\theta}^2\end{aligned}$$

The pendulum is under the influence of gravity:

$$U = mgl\left(1 - \cos\theta\right)$$

Where we have set the potential in the fully vertical position to be our zero point. We can now write out the Lagrangian:

$$\begin{aligned}\mathcal{L} &= T - U \\ &= \frac{1}{2}ml^2\dot{\theta}^2 - mgl\,(1-\cos\theta)\end{aligned}$$

Now writing out the Euler-Lagrange equation for θ:

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\theta}}\right) &= \frac{\partial\mathcal{L}}{\partial\theta} \\ \frac{d}{dt}\left(ml^2\dot{\theta}\right) &= mgl\sin\theta \\ \ddot{\theta} &= -\frac{g}{l}\sin\theta\end{aligned}$$

Which is the equation of motion for a simple pendulum, and a small angle approximation recovers simple harmonic motion.

Let us now discuss generalized coordinates for a constrained system. We say that the parameters $q_1, q_2, \ldots, q_n$ are generalized coordinates for a system of N particles, $\alpha = 1, 2, \ldots N$, with positions $\boldsymbol{r}_\alpha$, if each position $\boldsymbol{r}_\alpha$ can be represented as a function of $q_1, \ldots, q_n$, and vice versa, we can represent each q_i as some function of $\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_N$. The number of generalized coordinates n is the smallest number that allows the system to be parameterized this way. For an unconstrained system of particles in 3 dimensions, $n = 3N$. If the number of generalized coordinates is less than this, then the system is constrained.

For example, consider a double pendulum, which has 4 coordinates, x_1, y_1, x_2, y_2. However, we can condense this down to two coordinates, θ_1 and θ_2, which uniquely determine x_1, x_2, y_1, y_2.

If we have a system that is time dependent, such as a pendulum hung inside a moving box, then we can still represent the system using the generalized coordinate θ, but there will be some dependence on t. If a system can be represented using generalized coordinates with no dependence on t, then the coordinates are *natural* coordinates.

The number of *degrees of freedom* is the number of ways the system can move from any given initial configuration. In most simple examples, this is the same as the number of generalized coordinates needed to describe the system. These systems are called *holonomic*. For an example of a nonholonomic system, consider a plane, on which there is a groove, with a coin in the groove. The coin can only forwards or backwards, giving us 1 degree of freedom. We have a generalized coordinate, the angle relative to a chosen point on the rim of the coin, and the point contacting the ground. However, consider a ball on a plane. This has two degrees of freedom, but if we want to pick a reference point, we actually see that we need 3 generalized coordinates to specify a configuration (Euler angles?).

Let us now discuss Lagrange's equations for constrained systems. To keep things simple, let us consider a single particle constrained to move on a two dimensional surface. We have two generalized coordinates q_1 and q_2, which fully describe the system. Let us assume that there are two kinds of forces acting on the system. The first are the forces of constraint, which keep the particle on the surface, and therefore do no work, and the second are all other forces, such as gravity, which can do work. We denote the non constraint forces by $\boldsymbol{F}$, and the constraint forces by $\boldsymbol{F}_{\text{constr.}}$:

$$\boldsymbol{F}_{\text{total}} = \boldsymbol{F} + \boldsymbol{F}_{\text{constr.}}$$

We assume that the non-constraint forces are the gradient of some potential:

$$\boldsymbol{F} = -\nabla U\left(\boldsymbol{r}, t\right)$$

We write the Lagrangian of the system disregarding the constraints:

$$\mathcal{L} = T - U$$

And we will show that this will be the correct prescription for dealing with constrained systems.

Consider two points, $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$, through which the particle passes at times t_1 and t_2 respectively. Suppose the particle follows path $\boldsymbol{r}\left(t\right)$, and we have a infinitely close "wrong" path, $\boldsymbol{R}\left(t\right)$:

$$\boldsymbol{R}\left(t\right) = \boldsymbol{r}\left(t\right) + \boldsymbol{\varepsilon}\left(t\right)$$

Where $\boldsymbol{\varepsilon}\left(t\right)$ is zero at the endpoints, and is infinitesimal for all t.

We assume that both paths are constrained to the surface, and since they are infinitesimally close, the vector that joins them, $\boldsymbol{\varepsilon}\left(t\right)$, must also lie on the surface. This is trivially true for a plane, and due to the infinitesimal nature of the paths, this is true for all locally planar surfaces.

Now let us consider the action integral:

$$S = \int_{t_1}^{t_2} \mathcal{L}\, dt$$

We can compute the difference between the Lagrangians for the true path and the wrong path:

$$\Delta\mathcal{L} = m\dot{\boldsymbol{r}} \cdot \dot{\boldsymbol{\varepsilon}} - \boldsymbol{\varepsilon} \cdot \nabla U + \mathcal{O}\left(\varepsilon^2\right)$$

Working through this, we are left with

$$S = -\int dt\, \boldsymbol{\varepsilon} \cdot \underbrace{\left[m\ddot{\boldsymbol{r}} + \boldsymbol{\nabla} U\right]}_{\boldsymbol{F}_{\text{constr.}}}$$

Note that the quantity in the brackets is the constraint force. Since $\boldsymbol{\varepsilon}$ was previously shown to lie on the surface, and the constraint forces are perpendicular to the surface, this dot product must always be zero. The action, which we defined using only non-constraint forces, is stationary at the true path with respect to all other paths *on the surface*. Thus we have proved Hamilton's principle for paths consistent with the constraints. Although we showed this for a single particle, this holds for constrained systems with a general number of degrees of freedom and generalized coordinates.

This allows us to blindly apply the Euler-Lagrange equations to the generalized coordinates of the system, disregarding any constrained forces.

2.4.2 Examples

Let us do a simple example of Lagrangian mechanics. Consider Atwood's machine, two masses connected by a string pulled over a pulley. We define the distance from the center of the pulley to m_1 to be x, and the distance from the center of the pulley to m_2 to be y. The total length of the

string is then given by $l = x + y + \pi R$, where the radius of the pulley is R. The kinetic energy is given by

$$\begin{aligned} T &= \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2\dot{y}^2 \\ &= \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2\left(-\dot{x}\right)^2 \\ &= \frac{1}{2}\left(m_1 + m_2\right)\dot{x}^2 \end{aligned}$$

Computing the potential energy:

$$\begin{aligned} U &= -m_1 g x - m_2 g y \\ &= -m_1 g x + m_2 g x + \text{const.} \end{aligned}$$

Writing out the Lagrangian:

$$\begin{aligned} \mathcal{L} &= T - U \\ &= \frac{1}{2}\left(m_1 + m_2\right)\dot{x}^2 + \left(m_1 - m_2\right) g x \end{aligned}$$

Now writing out the Euler-Lagrange equations:

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial \mathscr{L}}{\partial \dot{x}}\right) - \frac{\partial \mathscr{L}}{\partial x} &= 0 \\ \left(m_1 + m_2\right)\ddot{x} - \left(m_1 - m_2\right) g &= 0 \end{aligned}$$

Which provides the standard result for the acceleration:

$$\ddot{x} = \frac{m_1 - m_2}{m_1 + m_2} g$$

Now let us consider a block sliding down a wedge, where the wedge can also slide on the ground. The inclination of the wedge is given by angle α.

In this case, we define two coordinates, q_1 is the distance from the top of the wedge to the mass sliding down the wedge (measured parallel to the slope of the wedge), and q_2 is the distance from the starting point to the edge of the wedge, measured parallel to the ground.

The kinetic energy of the wedge is given by

$$T_M = \frac{1}{2}M\dot{q}_2^2$$

To find the kinetic energy of the little mass, we first compute the x and y coordinates:

$$\begin{aligned} x_m &= q_2 + q_1 \cos\alpha \\ y_m &= -q_1 \sin\alpha \end{aligned}$$

Where we have discarded any other constants, since they won't end up mattering.The kinetic energy is then given by

$$T_m = \frac{1}{2}\left(\dot{x}_m^2 + \dot{y}_m^2\right)$$

$$
\begin{aligned}
&= \frac{1}{2}m\left(\dot{q}_2 + \dot{q}_1\cos\alpha\right)^2 + \frac{1}{2}m\left(-\dot{q}_1\sin\alpha\right)^2 \\
&= \frac{1}{2}m\left(\dot{q}_1^2\sin^2\alpha + \dot{q}_2^2 + \dot{q}_1^2\cos^2\alpha + 2\dot{q}_1\dot{q}_2\cos\alpha\right) \\
&= \frac{1}{2}m\left(\dot{q}_2^2 + \dot{q}_1^2 + 2\dot{q}_1\dot{q}_2\cos\alpha\right)
\end{aligned}
$$

The total kinetic energy is then given by

$$T = \frac{1}{2}M\dot{q}_2^2 + \frac{1}{2}m\left(\dot{q}_2^2 + \dot{q}_1^2 + 2\dot{q}_1\dot{q}_2\cos\alpha\right)$$

We can compute the potential energy in the system:

$$U = -mgq_1\sin\alpha$$

And then write out the Lagrangian:

$$
\begin{aligned}
\mathcal{L} &= T - U \\
&= \frac{1}{2}M\dot{q}_2^2 + \frac{1}{2}m\left(\dot{q}_2^2 + \dot{q}_1^2 + 2\dot{q}_1\dot{q}_2\cos\alpha\right) + mgq_1\sin\alpha
\end{aligned}
$$

Now applying the Euler-Lagrange equations (Eqn. 4):

$$
\begin{aligned}
\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{q}_2}\right) - \frac{\partial\mathcal{L}}{\partial q_2} &= 0 \\
\frac{d}{dt}\left(M\dot{q}_2 + m\dot{q}_2 + \dot{q}_1\cos\alpha\right) + 0 &= 0 \\
\left(M+m\right)\dot{q}_2 + m\dot{q}_1\cos\alpha &= \text{constant}
\end{aligned}
$$

We see that this implies that momentum is conserved along the x direction.

Now we can look at q_1:

$$
\begin{aligned}
\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{q}_1}\right) - \frac{\partial\mathcal{L}}{\partial q_1} &= 0 \\
\frac{d}{dt}\left(m\dot{q}_1 + m\dot{q}_2\cos\alpha\right) - mg\sin\alpha &= 0 \\
m\ddot{q}_1 + m\ddot{q}_2\cos\alpha &= mg\sin\alpha \\
\ddot{q}_1 + \ddot{q}_2\cos\alpha &= g\sin\alpha
\end{aligned}
$$

We have two differential equations of two variables, q_1 and q_2. We can now solve for q_1, by eliminating q_2 from this equations. From momentum conservation, we have that

$$\ddot{q}_2 = -\frac{m\cos\alpha}{M+m}\ddot{q}_1$$

Inserting this into the second diffeq:

$$
\begin{aligned}
\ddot{q}_1 - \left(\frac{m\cos^2\alpha}{M+m}\right)\ddot{q}_1 &= g\sin\alpha \\
\left(1 - \frac{m\cos^2\alpha}{M+m}\right)\ddot{q}_1 &= g\sin\alpha
\end{aligned}
$$

From this, we find that

$$\ddot{q}_1 = \frac{g \sin\alpha}{1 - \frac{m\cos^2\alpha}{M+m}}$$

Now let us take some limits, in order to check whether this agrees with what we expect. If we take the $M \gg m$ limit, we expect the wedge to not move as the block slides down the wedge. In this case, the acceleration reduces to the expected fixed wedge result, $\ddot{q}_1 = g\sin\alpha$.

If instead we take $\alpha \to \frac{\pi}{2}$, so we have a vertical edge, we obtain that $\ddot{q}_1 = g$, which is the expected freefall result.

Now let us consider a more difficult system. We have a bead on a hoop of radius R, which can rotate along its central axis. This problem is easiest to solve in spherical coordinates. We measure θ as the angle from the vertical to the bead, ρ as the horizontal distance from the rotation axis to the bead, and ϕ as the "inward" and "outward" angle.

The velocity in the ϕ direction is given by

$$\begin{aligned} v_\phi &= \rho\omega \\ &= R\omega\sin\theta \end{aligned}$$

And the velocity in the θ direction is given by

$$v_\theta = R\dot{\theta}$$

Thus the kinetic energy is given by

$$\begin{aligned} T &= \frac{1}{2}m\left(v_\phi^2 + v_\theta^2\right) \\ &= \frac{1}{2}m\left(R^2\dot{\theta}^2 + R^2\sin^2\theta\omega^2\right) \end{aligned}$$

We can measure the potential energy from the bottom of the hoop:

$$U = mgR\left(1 - \cos\theta\right)$$

We can then write out the Lagrangian:

$$\begin{aligned} \mathcal{L} &= T - U \\ &= \frac{1}{2}m\left(R^2\dot{\theta}^2 + R^2\sin^2\theta\omega^2\right) - mgR\left(1-\cos\theta\right) \end{aligned}$$

Now writing out the Euler-Lagrange equations (Eqn. 4):

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\theta}}\right) - \frac{\partial\mathcal{L}}{\partial\theta} &= 0 \\ \frac{d}{dt}\left(mR^2\dot{\theta}\right) - \left(mR^2\omega^2\sin\theta\cos\theta - mgR\sin\theta\right) &= 0 \\ mR^2\ddot{\theta} &= mR^2\omega^2\sin\theta\cos\theta - mgR\sin\theta \\ \ddot{\theta} &= \omega^2\sin\theta\cos\theta - \frac{g}{R}\sin\theta \end{aligned}$$

We want to find the equilibrium positions and the frequency of small oscillations. Equilibrium positions are cases where if $\dot{\theta} = 0$, then $\ddot{\theta}$ is also 0, there is no velocity and no acceleration, the bead just sits there.

For this problem, equilibrium points satisfy $\ddot{\theta} = 0$ (and we will insert the boundary condition that $\dot{\theta} = 0$ later), and thus

$$\left(\omega^2 \cos\theta - \frac{g}{R}\right) \sin\theta = 0$$

From this, we see that there are two possibilities. The first is that $\sin\theta = 0$, which gives $\theta = 0$ or $\theta = \pi$, which correspond to the bottom and top of the hoop. The second possibility is $\cos\theta = \frac{g}{R}\omega^2$. Now let us consider each of these, and determine whether they are stable or unstable, and if they are stable, the frequency of small oscillations. The first two make sense as equilibrium points, but what is the third case? Consider the frame of reference of the rotating hoop. In this frame, the bead has an outwards centrifugal force (pseudoforce generated by our noninertial reference frame), gravity pulling down, and the normal force of the wire. There will be some value of the angle for which the centrifugal and gravity forces cancel along the direction parallel to the wire, and the bead will not move.

Let us denote these equilibrium points as θ_0. Let us begin with the one at the top of the hoop, $\theta_0 = \frac{\pi}{2}$.

To decide whether this point is stable, let us consider a slight displacement from the equilibrium point. If the bead rolls back to the equilibrium point, then it is a stable equilibrium, otherwise it is an unstable equilibrium point.

Consider $\theta = \theta_0 - \varepsilon = \pi - \varepsilon$, a small displacement from the equilibrium. In this case, we can compute that $\cos(\pi - \varepsilon) = -1 + \mathcal{O}\left(\varepsilon^2\right)$, and $\sin(\pi - \varepsilon) = \varepsilon$. We can now insert these into the relation for $\ddot{\theta}$, which is equal to $\ddot{\varepsilon}$:

$$\ddot{\varepsilon} = \left(\omega^2 + \frac{g}{R}\right)\varepsilon$$

We note that the coefficient is positive, so ε increases with time, and thus θ increases with time. Thus the bead will move away from the equilibrium point, and thus the point is an unstable equilibrium, which we expected, placing the bead at the top of the hoop is intuitively not stable.

Now let us consider the case where $\theta_0 = 0$. In this case, $\theta = \theta_0 + \varepsilon = \varepsilon$:

$$\begin{aligned}\ddot{\varepsilon} &= \left(\omega^2 \cos(\varepsilon) - \frac{g}{R}\right)\sin(\varepsilon) \\ &= \left(\omega^2 - \frac{g}{R}\right)\varepsilon\end{aligned}$$

Now let us consider $\omega^2 - \frac{g}{R}$. If $\omega > \frac{g}{R}$, then ε grows with time, and the equilibrium is unstable. However, if $\omega < \frac{g}{R}$, then $\ddot{\varepsilon} < 0$, and the equilibrium is stable, the small shift will lead to the bead going back down to the bottom. Now let us find the frequency of small oscillations around this equilibrium point in the case where $\omega^2 - \frac{g}{R} < 0$.

We have that

$$\ddot{\varepsilon} + \left(\frac{g}{R} - \omega^2\right)\varepsilon = 0$$

Recall that this is the simple harmonic motion differential equation, with frequency:

$$f = \frac{1}{2\pi}\sqrt{\frac{g}{R} - \omega^2}$$

Now let us consider the final equilibrium point. In this case, $\cos\theta_0 = \frac{g}{R\omega^2}$, and we set $\theta = \theta_0 + \varepsilon$.

We can again compute $\ddot{\varepsilon}$:

$$\begin{aligned}
\ddot{\varepsilon} &= \left(\omega^2\cos\left(\theta_0+\varepsilon\right) - \frac{g}{R}\right)\sin\left(\theta_0+\varepsilon\right) \\
&= \left(\omega^2\left(\cos\theta_0 - \varepsilon\sin\theta_0\right) - \frac{g}{R}\right)\left(\sin\theta_0 + \varepsilon\cos\theta_0\right) \\
&= \left(\omega^2\cos\theta_0 - \varepsilon\omega^2\sin\theta_0 - \frac{g}{R}\right)\left(\sin\theta_0 + \varepsilon\cos\theta_0\right) \\
&= \omega^2\cos\theta_0\sin\theta_0 - \varepsilon\omega^2\sin^2\theta_0 + \varepsilon^2\omega^2\cos\theta_0\sin\theta_0 - \frac{g}{R}\sin\theta_0 + \varepsilon\frac{g}{R}\cos\theta_0 \\
&= \omega^2\cos\theta_0\sin\theta_0 - \varepsilon\left(\omega^2\sin^2\theta_0 + \frac{f}{R}\cos\theta_0\right) - \frac{g}{R}\sin\theta_0
\end{aligned}$$

Where we have applied:

$$\begin{aligned}
\cos\left(\theta_0+\varepsilon\right) &= \cos\theta_0\underbrace{\cos\varepsilon}_{1} - \sin\theta_0\underbrace{\sin\varepsilon}_{\varepsilon} \\
\sin\left(\theta_0+\varepsilon\right) &= \sin\theta_0\underbrace{\cos\varepsilon}_{1} + \cos\theta_0\underbrace{\sin\varepsilon}_{\varepsilon}
\end{aligned}$$

Note that this can be derived by using Euler's identity, $e^{iA} = \cos A + i\sin A$. We also discard terms of order ε^2. We then insert the fact that $\cos\theta_0 = \frac{g}{R\omega^2}$:

$$\begin{aligned}
\ddot{\varepsilon} &= \left(-\omega^2\sin^2\theta_0\right)\varepsilon \\
&= -\omega^2\left(1-\cos^2\theta_0\right)\varepsilon \\
&= -\omega^2\left(1 - \frac{g^2}{R^2\omega^4}\right)\varepsilon \\
&= \left(-\omega^2 + \frac{g^2}{R^2\omega^2}\right)\varepsilon
\end{aligned}$$

We thus have that

$$\ddot{\varepsilon} + \left(\omega^2 - \frac{g^2}{\omega^2R^2}\right)\varepsilon = 0$$

This is unstable if $\omega^2 - \frac{g^2}{\omega^2R^2} < 0$, which is true if $\frac{g}{\omega^2R} > 1$. Conversely, it will be stable if the opposite condition is true, $\frac{g}{\omega^2R} < 1$.

2.4.3 Forces of Constraint

Using Lagrangian methods, how can we calculate forces of constraint? This method uses the functional analogue of Lagrange multipliers.

Let us first review Lagrange multipliers.

What is the largest area that a rectangle can have if we have the constraint that the perimeter is 8?

Let us solve this in 3 different ways. The first method is to let the area be A, and the sides be of length a and b. The perimeter is given by $2(a+b)=8$, and $A=ab$. We want to extremize A subject to the constraint that $a+b=4$. We can do this by eliminating b, $b=4-a$, and then inserting this into the equation for the area, $A=a(4-a)$. We can then take the derivative and set it to 0:

$$\begin{aligned}\frac{dA}{da}&=0\\4-2a&=0\end{aligned}$$

Which gives us that $a=b=2$, and $A=4$.

The second method is to extremize A with respect to a, but take into account that b is not independent of a:

$$\begin{aligned}dA&=\left(\frac{\partial A}{\partial a}\right)da+\left(\frac{\partial A}{\partial b}\right)db\\\frac{dA}{da}&=\left(\frac{\partial A}{\partial a}\right)+\left(\frac{\partial A}{\partial b}\right)\left(\frac{\partial b}{\partial a}\right)\end{aligned}$$

We see that $\frac{db}{da}=-1$, and we have that

$$\frac{dA}{da}=b-a$$

Setting this equal to 0, we find that $a=b$, which gives us that $a=b=2$, and $A=4$.

Finally, the third method is to apply Lagrange multipliers. We first write the constraint equation as $a+b-4=0$. Now, rather than extremize A, we extremize $F=A+\lambda(a+b-4)=ab+\lambda(a+b-4)$. We extremize F with respect to a, b, and λ independently:

$$\begin{aligned}\frac{\partial F}{\partial a}&=0\to b+\lambda=0\\\frac{\partial F}{\partial b}&=0\to a+\lambda=0\\\frac{\partial F}{\partial \lambda}&=0\to a+b-4=0\end{aligned}$$

From the first two equations, we find that $a=b=-\lambda$, and inserting this into the third equation, we see that $a=b=2$.

Now consider Atwood's machine. We assume that the length of the string is constant:

$$x+y+\pi R=l$$

This has Lagrangian:

$$\mathcal{L}=\frac{1}{2}m_1\dot{x}^2+\frac{1}{2}m_2\dot{y}^2+m_1gx+m_2gy$$

Now, one way to deal with this would be to use the constraint on the length to eliminate y from the Lagrangian, and have it be fully in terms of x. We then extremize this with respect to $x(t)$, and

solve. This is what we did when we considered the Atwood's machine earlier. However, this does not easily provide the force of constraint, the tension in the string.

Let us now consider the analogue of method 2, extremizing $\mathcal{L}$ with respect to x, keeping in mind that y is not independent of x.

$$\delta S = -\int dt \left[\frac{d}{dt}\left(\frac{d\mathcal{L}}{d\dot{x}}\right) - \frac{\partial L}{\partial x}\right]\delta x + \left[\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{y}}\right) - \frac{\partial \mathcal{L}}{\partial y}\right]\delta y$$

From the constraint equation, $x + y + \pi R$ is constant, and thus $\delta x = -\delta y$:

$$\delta S = -\int dt \left[\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) - \frac{\partial \mathcal{L}}{\partial x} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{y}}\right) + \frac{\partial \mathcal{L}}{\partial y}\right]\delta x$$

Extremizing with respect to δx:

$$\left[\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) - \frac{\partial \mathcal{L}}{\partial x}\right] - \left[\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{y}}\right) - \frac{\partial \mathcal{L}}{\partial y}\right] = 0$$
$$(m_1\ddot{x} - m_1 g) - (m_2\ddot{y} - m_2 g) = 0$$

Now eliminating y using the constraint equation:

$$(m_1 + m_2)\,\ddot{x} = (m_1 - m_2)\,g$$

Which is the same equation of motion obtained from the first method. However, we again see that the force of constraint is not easy to obtain.

Let us now discuss the third method, the Lagrange multiplier method. This is the method that should be used when we care about deriving the force of constraint.

We write the constraint as

$$(x + y + \pi R - l) = 0$$

We now extremize the value:

$$\mathcal{L} + \lambda\,(x + y + \pi R - l)$$

This provides a new action:

$$\begin{aligned} S' &= \int dt\,\mathcal{L}' \\ &= \int dt\,\left[\mathcal{L} + \lambda\,(t)\,(x + y + \pi R - l)\right] \end{aligned}$$

This will provide us three equations, by extremizing with respect to x, y, and λ respectively:

$$\begin{aligned} m_1\ddot{x} - m_1 g - \lambda &= 0 \\ m_2\ddot{y} - m_2 g - \lambda &= 0 \\ x + y + \pi R - l &= 0 \end{aligned}$$

We now eliminate λ from the first two equations:

$$m_1\ddot{x} - m_1 g - m_2\ddot{y} + m_2 g = 0$$

We can then eliminate y using the constraint equation:

$$\ddot{x} = \frac{m_1 - m_2}{m_1 + m_2} g$$

How do we obtain the force of constraint? Consider the first equation that we obtained:

$$m_1\ddot{x} = m_1 g + \lambda$$

By Newton's Law, this is the total force acting on m_1. $m_1 g$ is the force of gravity, and therefore λ must represent the constraint force, the tension. We can solve for λ by inserting the expression for $\ddot{x}$:

$$\begin{aligned}\lambda &= m_1\ddot{x} - m_1 g \\ &= -2\frac{m_1 m_2}{m_1 + m_2} g\end{aligned}$$

Why does this method work in the general case? We will show that the Langrange multiplier method is basically equivalent to the second method we used. Consider a Lagrangian that is a function of two coordinates, q_1 and q_2, $\mathcal{L} = \mathcal{L}(q_1, q_2, \dot{q}_1, \dot{q}_2)$, where the two coordinates are not independent, but are related by some constraint $g(q_1, q_2) = 0$. We can obtain the equations of motion by using the second method:

$$\begin{aligned}\delta s &= -\int dt\, \delta\mathcal{L} \\ &= -\int dt \left[\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{q}_1}\right) - \frac{\partial\mathcal{L}}{\partial q_1}\right]\delta q_1 + \left[\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{q}_2}\right) - \frac{\partial\mathcal{L}}{\partial q_2}\right]\delta q_2 \\ &= -\int dt \left[\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{q}_1}\right) - \frac{\partial\mathcal{L}}{\partial q_1} + \left(\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{q}_2}\right) - \frac{\partial\mathcal{L}}{\partial q_2}\right)\frac{\delta q_2}{\delta q_1}\right]\delta q_1\end{aligned}$$

We require that this vanish for arbitrary q_1:

$$\left[\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{q}_1}\right) - \frac{\partial\mathcal{L}}{\partial q_1}\right] + \frac{\delta q_2}{\delta q_1}\left[\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{q}_2}\right) - \frac{\partial\mathcal{L}}{\partial q_2}\right] = 0$$

From the constraint equation $g(q_1, q_2) = 0$, we have that

$$\begin{aligned}\frac{\partial g}{\partial q_1}\delta q_1 + \frac{\partial g}{\partial q_2}\delta q_2 &= 0 \\ \frac{\delta q_2}{\delta q_1} &= -\frac{\frac{\partial g}{\partial q_1}}{\frac{\partial g}{\partial q_2}}\end{aligned}$$

We can now insert this into our previous equation:

$$\left[\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{q}_1}\right) - \frac{\partial\mathcal{L}}{\partial q_1}\right] - \frac{\frac{\partial g}{\partial q_1}}{\frac{\partial g}{\partial q_2}}\left[\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{q}_2}\right) - \frac{\partial\mathcal{L}}{\partial q_2}\right] = 0$$

We then eliminate q_2 in favor of q_1 using the constraint equation, and we obtain the equation of motion for q_1.

Now let us prove that this method is mathematically equivalent to the Lagrange multiplier method.

Using the Lagrange multiplier method:

$$S' = \int dt \left[\mathcal{L} + \lambda g\left(q_1, q_2\right)\right]$$

We then extremize this with respect to q_1,q_2, and λ, treating them as independent variables:

$$\begin{aligned}
\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_1}\right) - \frac{\partial \mathcal{L}}{\partial q_1} - \lambda \frac{\partial g}{\partial q_1} &= 0 \\
\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_2}\right) - \frac{\partial \mathcal{L}}{\partial q_2} - \lambda \frac{\partial g}{\partial q_2} &= 0 \\
g\left(q_1, q_2\right) &= 0
\end{aligned}$$

Eliminating λ from the first two equations, we see that we are left with the result from method 2:

$$\left[\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_1}\right) - \frac{\partial \mathcal{L}}{\partial q_1}\right] - \frac{\frac{\partial g}{\partial q_1}}{\frac{\partial g}{\partial q_2}}\left[\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_2}\right) - \frac{\partial \mathcal{L}}{\partial q_2}\right] = 0$$

We can again eliminate q_2 via the constraint equation to obtain the equation of motion for q_1.

How do we find the forces of constraint? Looking at the extremization with respect to q_1, we see that

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_1}\right) = \frac{\partial \mathcal{L}}{\partial q_1} + \lambda \frac{\partial g}{\partial q_1}$$

The first term is the generalized force, and the second term is the generalized force of constraint. However, we care about the actual force of constraint, not the generalized force of constraint. To determine the force of constraint, we assume that the generalized force of constraint is associated with a potential:

$$\lambda \frac{\partial g}{\partial q_1} = -\frac{\partial V_c}{\partial q_1}$$

Then the force of constraint is given by

$$\begin{aligned}
F_c &= -\nabla V_c \\
&= -\frac{\partial V_c}{\partial q_1} \nabla q_1 \\
&= \lambda \frac{\partial g}{\partial q_1} \nabla q_1
\end{aligned}$$

This will give the force of constraint for q_1. If q_2 describes the location of the same particle as q_1, then there is an additional contribution to the force of constraint:

$$F_c = \lambda \frac{\partial g}{\partial q_1} \nabla q_1 + \lambda \frac{\partial g}{\partial q_2} \nabla q_2$$

Consider a block of mass m sliding down a fixed ramp of angle θ. We want to find the normal force using the Lagrangian formalism. We choose y to be the vertical coordinate, and x to be the horizontal coordinate. We can write out the Lagrangian:

$$\mathcal{L} = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) - mgy$$

In these coordinates, the constraint is that the block must stay on the wedge, so $y = x\tan\theta$. We can then define the new Lagrangian:

$$\mathcal{L}' = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) - mgy + \lambda\left(y - x\tan\theta\right)$$

Now we obtain the equations of motion:

$$\begin{aligned} m\ddot{x} + \lambda\tan\theta &= 0 \\ m\ddot{y} + mg + \lambda &= 0 \end{aligned}$$

Eliminating λ:

$$m\ddot{x} + \left(m\ddot{y} + mg\right)\tan\theta = 0$$

We now apply the constraint to remove y:

$$m\ddot{x} + \frac{d^2}{dt^2}\left(x\tan\theta\right)\tan\theta + mg\tan\theta = 0$$

Working through this, we are left with

$$\begin{aligned} \ddot{x}\left(1 + \tan^2\theta\right) + g\tan\theta &= 0 \\ \ddot{x}\sec^2\theta &= -g\tan\theta \\ \ddot{x} &= -g\sin\theta\cos\theta \end{aligned}$$

From this, we have that $\ddot{y} = -g\sin^2\theta$. Writing out the acceleration as a vector:

$$\boldsymbol{a} = -g\sin\theta\begin{pmatrix}\cos\theta \\ \sin\theta\end{pmatrix}$$

The magnitude of the acceleration is $g\sin\theta$, down the slope. Now we want to find the normal force, so we can pick one of the equations of motion. We can rewrite λ:

$$\begin{aligned} \lambda &= mg + m\ddot{y} \\ &= mg\left(1 - \sin^2\theta\right) \\ &= mg\cos^2\theta \end{aligned}$$

We can write out the force of constraint:

$$\boldsymbol{F}_c = -\lambda\tan\theta\hat{x} + \lambda\hat{y}$$

$$= -mg\cos\theta \begin{pmatrix} \sin\theta \\ -\cos\theta \end{pmatrix}$$

Let us consider this same problem in a more convenient coordinate system. We have coordinate l, which is the distance down the wedge, and w, the perpendicular distance from the wedge. In this coordinate system, the kinetic energy is $\frac{1}{2}m\left(\dot{l}^2 + \dot{w}^2\right)$. The potential energy is given by $mgl\sin\theta + mgw\cos\theta$. This gives us Lagrangian:

$$\mathcal{L} = \frac{1}{2}m\left(\dot{l}^2 + \dot{w}^2\right) - mg\left(l\sin\theta + w\cos\theta\right)$$

Now we introduce the constraint, $w = 0$:

$$\mathcal{L}' = \frac{1}{2}m\left(\dot{l}^2 + \dot{w}^2\right) - mg\left(l\sin\theta + w\cos\theta\right) + \lambda w$$

The equations of motions obtained from this are:

$$\begin{aligned} m\ddot{l} + mg\sin\theta &= 0 \\ m\ddot{w} - \lambda + mg\cos\theta &= 0 \end{aligned}$$

From the second equation and the constraint equation, we have that $\lambda = mg\cos\theta$. From the first equation, we have that $\ddot{l} = -g\sin\theta$. The normal force can then be computed as:

$$\boldsymbol{F}_c = mg\cos\theta\hat{w}$$

Now let us consider a bead of mass m threaded through a very narrow rod of length l. The bead is free to slide without friction along the rod. The rod is fixed at one end, but is free to rotate in the horizontal plane about the fixed end. If the rod is rotating counterclockwise with constant angular velocity ω, find the equations of motion for the bead, and the force exerted by the rod on the bead. Neglect gravity.

Looking at the rod from a top-down perspective, we have two coordinates, the angle of the rod with a set horizontal position, ϕ, and the location of the bead on the rod, ρ. The angle is related to the angular velocity, $\phi = \omega t$. We can write out the Lagrangian:

$$\mathcal{L} = \frac{1}{2}m\left(\dot{\rho}^2 + \rho^2\dot{\phi}^2\right)$$

The modified Lagrangian is given by

$$\mathcal{L}' = \frac{1}{2}m\left(\dot{\rho}^2 + \rho^2\dot{\phi}^2\right) + \lambda\left(\phi - \omega t\right)$$

This produces the equations:

$$\begin{aligned} m\ddot{\rho} - m\rho\dot{\phi}^2 &= 0 \\ \frac{d}{dt}\left(m\rho^2\dot{\phi}\right) &= \lambda \end{aligned}$$

Now noting that $\dot{\phi} = \omega$:

$$m\ddot{\rho} - m\rho\omega^2 = 0$$

$$\frac{d}{dt}\left(m\rho^2\omega\right) = \lambda$$

The first equation simplifies down to

$$\ddot{\rho} = \rho\omega^2$$

This is a linear equation, and has solutions that are hyperbolic sines and cosines:

$$\rho\left(t\right) = A\cosh\left(\omega t\right) + B\sinh\left(\omega t\right)$$

Where A and B are determined by the initial conditions. The second equation of motion that we have gives us

$$2m\rho\omega^2\dot{\rho} = \lambda$$

Now noting that $\lambda = -\frac{\partial V_c}{\partial \phi}$, the potential for the force of constraint:

$$\begin{aligned}\boldsymbol{F}_c &= -\nabla V_c \\ &= -\frac{\partial V_c}{\partial \phi}\nabla\phi \\ &= +\frac{1}{\rho}\left[2m\rho\dot{\rho}\omega\right]\hat{\phi} \\ &= 2m\dot{\rho}\omega\hat{\phi}\end{aligned}$$

2.4.4 Two Body Central Force Problem

Consider two particles with masses m_1 and m_2, located at $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$. The only force acting on them is from their mutual interaction, which is assumed to be conservative, so it can be derived from a potential. Translational invariance implies that the force depends only on their relative positions:

$$U\left(\boldsymbol{r}_1, \boldsymbol{r}_2\right) = U\left(\boldsymbol{r}_1 - \boldsymbol{r}_2\right)$$

If a conservative force is central, then

$$U\left(\boldsymbol{r}_1 - \boldsymbol{r}_2\right) = U\left(\left|\boldsymbol{r}_1 - \boldsymbol{r}_2\right|\right)$$

We only care about the magnitude of the distance, and the force is along the line joining the two particles. It is thus convenient to introduce a relative position coordinate:

$$\boldsymbol{r} = \boldsymbol{r}_1 - \boldsymbol{r}_2$$

In which case the potential is just dependent on the magnitude of $\boldsymbol{r}$:

$$\mathcal{L} = \frac{1}{2}m_1\dot{\boldsymbol{r}}_1^2 + \frac{1}{2}m_2\dot{\boldsymbol{r}}_2^2 - U\left(\left|\boldsymbol{r}\right|\right)$$

What generalized coordinates should we use? So far, we have 9 coordinates, 3 for each of $\boldsymbol{r}_1$, $\boldsymbol{r}_2$, and $\boldsymbol{r}$. Instead, we consider the 'center of mass':

$$\boldsymbol{R} = \frac{m_1\boldsymbol{r}_1 + m_2\boldsymbol{r}_2}{m_1 + m_2}$$

These are the two coordinates that we use, $\boldsymbol{R}$ and $\boldsymbol{r}$. We define a total mass $m = m_1 + m_2$:

$$\boldsymbol{R} = \boldsymbol{r}_2 + \frac{m_1 \boldsymbol{r}}{m}$$

Note that the vector $\boldsymbol{R}$ points along the line joining the two particles.

Why is $\boldsymbol{R}$ a good choice of coordinates? We take advantage of the conservation of momentum:

$$\begin{aligned} \frac{d}{dt}\left(m_1 \dot{\boldsymbol{r}}_1 + m_2 \dot{\boldsymbol{r}}_2\right) &= 0 \\ \left(m_1 \dot{\boldsymbol{r}}_1 + m_2 \dot{\boldsymbol{r}}_2\right) &= \boldsymbol{p} \end{aligned}$$

Where the momentum $\boldsymbol{p}$ is constant. From this, we find that the center of mass moves with constant velocity:

$$\begin{aligned} \frac{d}{dt}\left(M\boldsymbol{R}\right) &= \boldsymbol{p} \\ \dot{\boldsymbol{R}} &= \text{constant} \end{aligned}$$

The system behaves as if the total mass was entirely located at the center of mass, and the center of mass is moving at a constant velocity. It is straightforward to choose coordinates such that the center of mass is not moving. In this frame, known as the center of mass frame or the center of momentum frame, the problem is especially simple.

We can write out $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ in terms of the center of mass position, and then write out the kinetic energy:

$$\begin{aligned} T &= \frac{1}{2}\left(m_1 \dot{\boldsymbol{r}}_1 + m_2 \dot{\boldsymbol{r}}_2\right) \\ &= \frac{1}{2} M \dot{\boldsymbol{R}} + \frac{1}{2} M \frac{m_1 m_2}{M^2} \dot{\boldsymbol{r}}^2 \end{aligned}$$

We define a parameter μ known as the reduced mass:

$$\begin{aligned} \mu &= \frac{m_1 m_2}{M} \\ &= \frac{m_1 m_2}{m_1 + m_2} \end{aligned}$$

Using this, we have that

$$T = \frac{1}{2} M \dot{\boldsymbol{R}}^2 + \frac{1}{2} \mu \dot{\boldsymbol{r}}^2$$

We see that we split up the kinetic energy into two parts, the kinetic energy of the center of mass, and the kinetic energy from the relative motion of the two particles. From the kinetic energy, we now consider the Lagrangian:

$$\begin{aligned} \mathcal{L} &= \frac{1}{2} M \dot{\boldsymbol{R}}^2 + \frac{1}{2} \mu \dot{\boldsymbol{r}}^2 - U\left(|\boldsymbol{r}|\right) \\ &= \mathcal{L}_{\text{CoM}} + \mathcal{L}_{\text{relative}} \end{aligned}$$

We see that the Lagrangian nicely splits into two separate Lagrangians, which, since there is no interaction between them, we can solve them separately. Also note that $\boldsymbol{R}$ is a *cyclic* coordinate,

it only appears in the Lagrangian in the form of its derivatives. When we write the equation of motion for a cyclic coordinate q, we will find that $\frac{\partial \mathcal{L}}{\partial \dot{q}}$ is a constant. In this case, $\boldsymbol{R}$ being a cyclic coordinate indicates that total momentum is conserved.

For the center of mass motion:

$$m\ddot{\boldsymbol{R}} = 0$$

Which we already knew. Looking at the relative motion:

$$\mathcal{L}_{\text{relative}} = \frac{1}{2}\mu\dot{\boldsymbol{r}}^2 - U\left(|\boldsymbol{r}|\right)$$

This is the same as the Lagrangian of a particle of mass μ moving in a central potential. From this, we find that

$$\mu\ddot{\boldsymbol{r}} = -\nabla U\left(|\boldsymbol{r}|\right)$$

In the case where we are in the center of mass frame, then we can throw out the center of mass term entirely, and we have reduced the problem to a 1 body problem, where we keep track of the relative position of the two particles, rather than both of their positions. In this frame, the two masses must have equal and opposite momenta.

Let us consider the conservation of angular momentum. In the center of mass frame, we have that

$$\begin{aligned}\boldsymbol{L} &= \boldsymbol{r}_1 \times \boldsymbol{p}_1 + \boldsymbol{r}_2 \times \boldsymbol{p}_2 \\ &\vdots \\ &= \mu\boldsymbol{r} \times \dot{\boldsymbol{r}}\end{aligned}$$

In the center of mass frame, we can think of this as again being a particle of mass μ moving in a central potential. Since the total angular momentum is conserved, then

$$\boldsymbol{r} \times \dot{\boldsymbol{r}} = \text{constant}$$

This is a vector perpendicular to both $\boldsymbol{r}$ and $\dot{\boldsymbol{r}}$. We note that this vector will always be perpendicular to the plane that $\boldsymbol{r}$ and $\dot{\boldsymbol{r}}$ share, and therefore, since it is constant, the motion of the system must be restricted to the plane.

We began with 6 coordinates, brought it down to 3 via the center of mass, and now we have planar motion, which brings us down to 2 coordinates, we can disregard any motion in the direction perpendicular to the plane.

Now that we know the motion is planar, we can work in cylindrical coordinates (where $z = 0$):

$$\mathcal{L} = \frac{1}{2}\mu\dot{\boldsymbol{r}}^2 + \frac{1}{2}\mu r^2\dot{\phi}^2 - U\left(r\right)$$

Since $\mathcal{L}$ doesn't explicitly depend on ϕ, it is a cyclic coordinate, and thus

$$\frac{d}{dt}\left(\mu r^2\dot{\phi}\right) = 0$$

And thus $\mu r^2\dot{\phi}$ is constant. This is the statement of conservation of angular momentum:

$$\mu r^2\dot{\phi} = L$$

From here, we finally have to start working with nontrivial equations. We can use the radial Euler-Lagrange :

$$\mu\ddot{\boldsymbol{r}} = -\frac{dU}{dr} + mr\dot{\phi}^2$$

Now substituting in the relation between angular momentum and $\dot{\phi}$:

$$\mu\ddot{\boldsymbol{r}} = -\frac{dU}{dr} + \frac{L^2}{\mu r^3}$$

If we can solve this for r, then we are done. This is the equation of motion of a particle of mass μ moving some potential that depends on a single variable. We see that this is a massive simplification compared to the original problem setup. In particular, this has the form of Newton's Second Law for a particle in one dimension subject to the actual force $-\frac{dU}{dr}$ and a fictitious centrifugal force

$$F_{cf} = \frac{L^2}{\mu r^3}$$

This centrifugal force can be obtained from a "centrifugal potential":

$$\begin{aligned} F_{cf} &= -\frac{d}{dr}\left(\frac{L^2}{2\mu r^2}\right) \\ &= -\frac{d}{dr}U_{cf} \end{aligned}$$

The radial equation then becomes

$$\mu\ddot{\boldsymbol{r}} = -\frac{d}{dr}\underbrace{\left(\frac{L^2}{2\mu r^2} + U(r)\right)}_{U_{\text{effective}}}$$

Now note that we have a particle in one dimension moving under the action of a conservative force. This leads to conservation of energy. We have Newton's Second Law:

$$\mu\ddot{r} = -\frac{dU_{\text{eff}}}{dr}$$

Multiplying by $\dot{r}$:

$$\mu\dot{r}\ddot{r} = -\dot{r}\frac{dU_{\text{eff}}}{dr}$$

Now integrating both sides:

$$\begin{aligned} \int \mu\dot{r}\ddot{r}\,dt &= -\int \frac{dr}{dt}\frac{dU_{\text{eff}}}{dr}\,dt \\ \frac{1}{2}\mu\dot{r}^2 &= U_{\text{eff}}(r) + E \\ \frac{1}{2}\mu\dot{r}^2 + U_{\text{eff}}\left(r\right) &= E \end{aligned}$$

We see that the total energy is a constant, denoted E. From here, we can solve this relation for $\dot{r}$:

$$\dot{r} = \sqrt{\frac{2}{\mu}\left(E - U_{\text{eff}}\left(r\right)\right)}$$

We can then integrate this with respect to t in order to find $r(t)$, and we will have solved the problem. There are only a few analytically solvable cases of this problem, but we see that we have simplified this greatly.

Let us consider this problem in the case of an inverse square law potential:

$$\begin{aligned} U(r) &= -G\frac{m_1 m_2}{r} \\ &= -\frac{\gamma}{r} \end{aligned}$$

In this case, the effective potential becomes

$$U_{\text{eff}} = -\frac{\gamma}{r} + \frac{l^2}{2\mu r^2}$$

We see that the first term is always negative, goes to 0 as $r \to \infty$, and goes to $-\infty$ as $r \to 0$. The second term is always positive, and has the opposite behavior near 0, $U_{\text{cf}} \to \infty$. Very close to 0, the centrifugal term wins, and so we go to ∞ near $r \to 0$, but we have a well for r slightly larger than zero, where the $1/r$ potential wins out over the $1/r^2$ potential.

We see that this potential has unbound orbits, where the energy of the incoming particle is positive, the particle approaches the origin and then reflects back out to ∞. The bound orbits have negative energy, and get trapped in the well.

Suppose we have a bound state, which oscillates between $r = r_{\min}$ and $r = r_{\max}$, inside the well. These points correspond to points where $\dot{r} = 0$, the energy is equal to the potential:

$$U_{\text{eff}}(r) = E$$

Also note that the case where E is exactly equal to U_{eff} at the bottom of the well, corresponds to circular motion, the radius between the two objects does not change.

Since $\dot{\phi} = L/\mu r^2$, as r changes, ϕ is also changing. For bound orbits, r oscillates between $r_{\min}$ and $r_{\max}$, but we never switch directions in the orbit, $\dot{\phi}$ always maintains the same sign.

2.5 Noether's Theorem

Theorem 2.1. ***Noether's Theorem.*** *Corresponding to every continuous symmetry transformation of the Lagrangian, there exists a conserved current.*

Conservation of energy, charge, and momentum, all can be derived from symmetries in the Lagrangian. For example, the $U(1)$ symmetry of QED leads to conservation of charge.

Consider Newton's Law:

$$\boldsymbol{F} = m\boldsymbol{a}$$

Which can be written in component form:

$$\begin{cases} F_x &= ma_x \\ F_y &= ma_y \\ F_z &= ma_z \end{cases}$$

The benefit of writing it in vector form is that we define how $\boldsymbol{F}$ and $\boldsymbol{a}$ transform under rotations. We say that both sides of the equation change covariantly under rotations, they change in the same way given a rotation of the coordinate system. Similarly, the moment of inertia is related to the angular moment and the angular velocity:

$$\boldsymbol{L} = \boldsymbol{I} \cdot \boldsymbol{\omega}$$

In this case, $\boldsymbol{L}$ and $\boldsymbol{\omega}$ are vectors, and $\boldsymbol{I}$ is a tensor.

In the case of electrodynamics, we can write Maxwell's equations as:

$$\begin{aligned} \partial_\mu F^{\mu\nu} &= j^\mu \\ \varepsilon^{\mu\nu\lambda\sigma} \partial_\nu F_{\lambda_\sigma} &= 0 \end{aligned}$$

Which is invariant under Lorentz transformations.

Let us prove Noether's theorem for a few symmetries. First, let us show that momentum conservation is a consequence of translation symmetry.

The laws of physics are invariant under translation, if we move a system from point a to point b, the system should behave the same way. By Noether's theorem, there must be a corresponding conserved current. We will show that the conserved current leads to momentum conserved.

Consider a system of particles with pairwise interactions in one dimension. The Lagrangian is given by

$$\mathcal{L} = \frac{1}{2} \sum_i m_i \dot{x}_i^2 + \sum_{i,j} V\left(x_i - x_j\right)$$

Consider a change of coordinates $x_i \to x_i' = x_i - a$ (where a is infinitesimal[2]), we shift all the coordinates by the same amount. This is a symmetry transformation:

$$\begin{aligned} \frac{dx_i'}{dt} &= \frac{d}{dt}\left(x_i - a\right) \\ &= \dot{x}_i \end{aligned}$$

Thus the Lagrangian in the new coordinates is just:

$$\begin{aligned} \mathcal{L}' &= \frac{1}{2} \sum_i m_i \dot{x}_i'^2 + \sum_{i,j} V\left(x_i' - x_j'\right) \\ &= \frac{1}{2} \sum_i m_i \dot{x}_i^2 + \sum_{i,j} V\left(x_i - x_j\right) \end{aligned}$$

We see that nothing changes, the Lagrangian is "form-invariant" under this translation. In particular, this implies that the action is also the same:

$$\int dt\, \mathcal{L}\left(x_i, \dot{x}_i\right) = \int dt\, \mathcal{L}\left(x_i', \dot{x}_i'\right)$$

The action is "scale-invariant" under this transformation. This is the main condition that we choose to determine symmetries in a system. In this case, we see that translation is a symmetry of the theory.

[2]Required because we need the symmetry to be continuous, Noether's theorem has no discrete analogue.

Now let us consider the difference between the Lagrangian in both coordinates, which is zero by form invariance:

$$\mathcal{L}\left(x_i', \dot{x}_i'\right) - \mathcal{L}\left(x_i, \dot{x}_i\right) = 0$$

Which (for infinitesimal a), can be written as

$$\sum_i \frac{\partial \mathcal{L}}{\partial x_i}\delta x_i + \sum_i \frac{\partial \mathcal{L}}{\partial \dot{x}_i}\delta x_i = 0$$

Now we note that $\delta x_i = x_i' - x_i = a$, and $\delta \dot{x}_i = 0$. Thus we have that

$$\sum_i \frac{\partial \mathcal{L}}{\partial x_i} = 0$$

At this point, we insert the equations of motion (Noether's theorem only applies to the classical path, which means that we have to constrain everything with the equations of motion):

$$\begin{aligned}
\sum_i \frac{\partial \mathcal{L}}{\partial x_i} &= 0 \\
\sum_i \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}_i}\right) &= 0 \\
\frac{d}{dt}\underbrace{\left(\sum_i \frac{\partial \mathcal{L}}{\partial \dot{x}_i}\right)}_{\text{total momentum}} &= 0
\end{aligned}$$

We see that by definition, this is the time derivative of the total momentum, and we have extracted that the total momentum is conserved. Anytime that we have a theory that is translationally invariant, we will always extract momentum conservation, and thus all theories that we write down should have translational invariance.

Let us now consider energy conservation. We will show that this arises from time translation invariance, the laws of physics are the same at all times.

Consider a Lagrangian $L\left[x\left(t\right), \dot{x}\left(t\right)\right]$ that does not depend explicitly on time, any time dependence is through the fact that x and $\dot{x}$ depend on time. Consider a change of coordinate $t \to t' = t + \varepsilon$, where ε is infinitesimal.

If we do this change of coordinates, we have that $x\left(t\right) = x\left(t' - \varepsilon\right) = x'\left(t'\right)$, where we define a new function x'. For constant ε:

$$\begin{aligned}
\frac{dx}{dt} &= \frac{dx'}{dt} \\
&= \frac{dx'}{dt'}\frac{dt'}{dt} \\
&= \frac{dx'}{dt'}
\end{aligned}$$

Then we can consider the Lagrangian:

$$\mathcal{L}\left[x\left(t\right), \dot{x}\left(t\right)\right] = \mathcal{L}\left[x'\left(t'\right), \dot{x}'\left(t'\right)\right]$$

We see that once again the Lagrangian is form-invariant. We can look at the action integral:

$$\int_{t_A}^{t_B} dt\, \mathcal{L}\left[x\left(t\right), \dot{x}\left(t\right)\right] = \int_{t_A+\varepsilon}^{t_B+\varepsilon} dt'\, \mathcal{L}\left[x'\left(t'\right), \dot{x}'\left(t'\right)\right]$$

The only difference here is the limits, which doesn't matter. The functional form of the integrals are the same, which gives us scale invariance. Thus we see that time translation is a symmetry of the system.

Let us now subtract the left hand side from the right hand side:

$$\int_{t_B}^{t_B+\varepsilon} dt'\, \mathcal{L}\left[x'\left(t'\right), \dot{x}'\left(t'\right)\right] + \int_{t_A+\varepsilon}^{t_A} dt'\, \mathcal{L}\left[x'\left(t'\right), \dot{x}'\left(t'\right)\right] + \int_{t_A}^{t_B} dt\, \left(\mathcal{L}\left[x'\left(t\right), \dot{x}'\left(t\right)\right] - \mathcal{L}\left[x\left(t\right), \dot{x}\left(t\right)\right]\right) = 0$$

What we have done is broken the right hand side into 3 integrals, an integral from t_A to t_B, an integral from t_B to $t_B + \varepsilon$, and an integral from t_A to $t_A + \varepsilon$. For the integrals over the interval of size ε, we argue that the Lagrangian doesn't have enough time to change by very much, and thus

$$\int_{t_B}^{t_B+\varepsilon} dt'\, \mathcal{L}\left[x'\left(t'\right), \dot{x}'\left(t'\right)\right] = \varepsilon \mathcal{L}\left[x'\left(t_B\right), \dot{x}'\left(t_B\right)\right]$$

And similarly for the other integral of interval size ε. We also rewrite the third integral, using a similar form as the case of momentum conservation:

$$\varepsilon \mathcal{L}\left[x'\left(t_B\right), \dot{x}'\left(t_B\right)\right] - \varepsilon \mathcal{L}\left[x'\left(t_A \dot{x}'\left(t_A\right)\right)\right] + \int_{t_A}^{t_B} dt\, \left[\frac{\partial \mathcal{L}}{\partial x}\delta x + \frac{\partial \mathcal{L}}{\partial \dot{x}}\delta \dot{x}\right] = 0$$

Now we can compute δx and $\delta \dot{x}$:

$$\begin{aligned}
\delta x &= x'\left(t\right) - x\left(t\right) \\
&= x\left(t-\varepsilon\right) - x\left(t\right) \\
&= -\varepsilon \dot{x} \\
\delta \dot{x} &= \dot{x}'\left(t\right) - \dot{x}\left(t\right) \\
&= -\varepsilon \frac{d}{dt}\dot{x}
\end{aligned}$$

Where we have Taylor expanded $x\left(t-\varepsilon\right)$ in order to produce the $\dot{x}$. We now replace the x's in the first two terms with xs, which we can do because the change is of order ε, and the two terms are already of order ε, so we disregard the $\mathcal{O}\left(\varepsilon^2\right)$ changes. We also insert our expressions for δx and $\delta \dot{x}$ into the third term:

$$\varepsilon \mathcal{L}\left[x\left(t_B\right), \dot{x}\left(t_B\right)\right] - \varepsilon \mathcal{L}\left[x\left(t_A\right), \dot{x}\left(t_A\right)\right] + \int_{t_A}^{t_B} dt\, \left[\frac{\partial \mathcal{L}}{\partial x}\left(-\varepsilon \dot{x}\right) + \frac{\partial \mathcal{L}}{\partial \dot{x}}\left(-\varepsilon \frac{d}{dt}\dot{x}\right)\right] = 0$$

Now we combine the first two terms into an integral:

$$\varepsilon \mathcal{L}\left[x\left(t_B\right), \dot{x}\left(t_B\right)\right] - \varepsilon \mathcal{L}\left[x\left(t_A\right), \dot{x}\left(t_A\right)\right] = \varepsilon \int_{t_A}^{t_B} dt\, \frac{d\mathcal{L}}{dt}$$

For the last term, we now insert the equation of motion:

$$\frac{\partial \mathcal{L}}{\partial x} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}}$$

And thus we have that

$$\varepsilon \int_{t_A}^{t_B} dt\, \frac{\partial \mathcal{L}}{\partial t} + \int_{t_A}^{t_B} dt\, \left(-\varepsilon \dot{x}\left(\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right)\right) - \varepsilon \frac{\partial \mathcal{L}}{\partial \dot{x}}\frac{d}{dt}\dot{x}\right) = 0$$

$$\varepsilon \int_{t_A}^{t_B} dt\, \frac{\partial \mathcal{L}}{\partial t} + \int_{t_A}^{t_B} dt\, \frac{d}{dt}\left(-\varepsilon \dot{x}\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) = 0$$

Where we notice that the second integrand is a total derivative. We can now merge everything:

$$\varepsilon \int_{t_A}^{t_B} dt\, \frac{d}{dt}\left[\mathcal{L} - \dot{x}\frac{\partial \mathcal{L}}{\partial \dot{x}}\right] = 0$$

The limits of integration here are arbitrary, so the integrand must be zero for all choices of time. Thus we have that

$$\frac{d}{dt}\left(\mathcal{L} - \dot{x}\frac{\partial \mathcal{L}}{\partial x}\right) = 0$$

Now we notice that this is just the definition of the Hamiltonian (up to a sign), which is the total energy in the system:

$$\frac{d}{dt}\mathcal{H} = 0$$

And thus we have conservation of energy.

2.6 Hamiltonian Mechanics

2.6.1 Hamilton's Equations

The basis of Lagrangian mechanics is the Lagrangian:

$$\mathcal{L} = T - U$$

which is the function of the generalized coordinates and their time derivatives. The coordinates specify the configuration of the system at a particular time, which can be thought of as defining a point in an n dimensional configuration space. The $2n$ coordinates $(q_1, q_2, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n)$ define a point in state space, and specify a set of initial conditions that determine a unique solution of the n second order differential equations, Lagrange's equations.

In Hamiltonian dynamics, the central role is played by the Hamiltonian $\mathcal{H}$, defined by

$$\mathcal{H} = \sum_i p_i \dot{q}_i - \mathcal{L}$$

The equations of motion involve derivatives of $\mathcal{H}$ rather than derivatives of $\mathcal{L}$. If the Lagrangian is time independent, as we just saw, $\mathcal{H}$ is constant. Just as the $2n$ coordinates in the Lagrangian case define a point in state space, the $2n$ coordinates in the Hamiltonian determine a unique point

in phase space. Hamilton's equations determine a path in phase space, starting from some initial configuration.

Let us derive Hamilton's equations for a 1 dimensional system.

We begin with the Hamiltonian:

$$\mathcal{H} = p\dot{q}\left(q,p\right) - \mathcal{L}\left(q,\dot{q}\left(q,p\right)\right)$$

Here $\dot{q}$ is determined implicitly from the definition of p:

$$\frac{\partial \mathcal{L}}{\partial \dot{q}} = p \rightarrow \dot{q}\left(q,p\right)$$

Now we can take the derivative of the Hamiltonian with respect to position, holding p fixed:

$$\begin{aligned}\left(\frac{\partial \mathcal{H}}{\partial q}\right)_p &= p\left(\frac{\partial \dot{q}}{\partial q}\right)_p - \left[\left(\frac{\partial \mathcal{L}}{\partial q}\right)_{\dot{q}} + \underbrace{\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right)_q}_{p}\left(\frac{\partial \dot{q}}{\partial q}\right)_p\right] \\ &= -\left(\frac{\partial \mathcal{L}}{\partial q}\right)_{\dot{q}} \\ &= -\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) \\ &= -\frac{dp}{dt}\end{aligned}$$

Where we have inserted the Euler-Lagrange's equation (Eqn 4). This is the first of Hamilton's equations. Now let us consider the derivative with respect to p, holding q fixed:

$$\begin{aligned}\left(\frac{\partial \mathcal{H}}{\partial p}\right)_q &= \dot{q}\left(q,p\right) + p\left(\frac{\partial \dot{q}}{\partial p}\right)_q - \left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right)_q\left(\frac{\partial \dot{q}}{\partial p}\right)_q \\ &= \dot{q}\end{aligned}$$

This gives us the second of Hamilton's equations. Together, we have that

$$\left(\frac{\partial \mathcal{H}}{\partial p}\right)_q = \dot{q} \tag{5}$$

$$\left(\frac{\partial \mathcal{H}}{\partial q}\right)_p = -\dot{p} \tag{6}$$

In the Lagrangian approach, for a system with one degree of freedom, we obtain a single second order equation of motion, while in the Hamiltonian approach, we obtain 2 first order equations of motion.

Let us do an example. Consider the Atwood's machine.

The Lagrangian for this system is given by:

$$\mathcal{L} = \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2\dot{y}^2 + m_1gx + m_2gy$$

Which we can rewrite as

$$\mathcal{L} = \frac{1}{2}(m_1 + m_2)\dot{x}^2 + (m_1 - m_2)gx$$

Now we can solve for p:

$$p = \frac{\partial \mathcal{L}}{\partial \dot{x}} = (m_1 + m_2)\dot{x}$$

We can now eliminate $\dot{x}$ in favor of p in the Lagrangian:

$$\mathcal{L} = \frac{1}{2}\frac{p^2}{m_1 + m_2} + (m_1 - m_2)gx$$

The Hamiltonian is then:

$$\begin{aligned}\mathcal{H} &= p\dot{x} - \mathcal{L} \\ &= \frac{p^2}{m_1 + m_2} - \mathcal{L} \\ &= \frac{1}{2}\frac{p^2}{m_1 + m_2} - (m_1 - m_2)gx\end{aligned}$$

We can then compute Hamilton equations:

$$\begin{aligned}\frac{\partial \mathcal{H}}{\partial x} &= -\dot{p} \\ \dot{p} &= (m_1 - m_2)g\end{aligned}$$

And the other equation gives:

$$\begin{aligned}\frac{\partial \mathcal{H}}{\partial p} &= \dot{x} \\ \dot{x} &= \frac{p}{m_1 + m_2}\end{aligned}$$

From this, we have that

$$\ddot{x} = \frac{m_1 - m_2}{m_1 + m_2}g$$

Which is exactly what we expect.

Why is the Hamiltonian formalism worth using if it gets the same result as the Lagrangian formalism? Looking at Hamilton's equations, we see that they have a symmetry built into them, known as a symplectic symmetry.

$$\begin{aligned}\frac{\partial \mathcal{H}}{\partial p} &= \dot{q} \\ \frac{\partial \mathcal{H}}{\partial q} &= -\dot{p}\end{aligned}$$

This symplectic structure underlies all of classical mechanics. Another benefit is that the Hamiltonian formalism allows for coordinate transformations to be done much more easily. We also are guaranteed unitarity, but we lose Lorentz invariance, while the Lagrangian loses unitarity, but maintains Lorentz

invariance. Each method has its pros and cons when applying the formalisms to quantum mechanics and quantum field theory.

Let us now consider Hamilton's equations when we have more than 1 degree of freedom. Suppose the configuration of the system is described by n coordinates, $q_1, q_2, \ldots, q_n$. The corresponding generalized velocities are $\dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n$. The Langrangian will be a function of these, $\mathcal{L}(q_i, \dot{q}_i, t)$. The generalized momenta are given by $p_i = \left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right)$, and the Hamiltonian is given by

$$\mathcal{H}(q_i, p_i, \mathcal{L}) = \sum_i p_i \dot{q}_i - \mathcal{L}$$

$\mathcal{H}$ is a function of n generalized coordinates and n generalized momenta. By following the same procedure as the 1 degree of freedom case, we find Hamilton's equations for a system with n degrees of freedom:

$$\dot{q}_i = \frac{\partial \mathcal{H}}{\partial p_i}$$
$$\dot{p}_i = -\frac{\partial \mathcal{H}}{\partial q_i}$$

These are $2n$ first-order differential equations.

Let us do an example. Consider a particle in a central force field. Recall that in polar coordinates, the kinetic and potential energies are given by

$$T = \frac{1}{2} m \left(\dot{r}^2 + r^2 \dot{\phi}^2\right)$$
$$U = U(r)$$

To solve this, we first write down the Lagrangian:

$$\begin{aligned} \mathcal{L} &= T - U \\ &= \frac{1}{2} m \left(\dot{r}^2 + r^2 \dot{\phi}^2\right) - U(r) \end{aligned}$$

Now we compute the conjugate momenta, p_r and p_ϕ:

$$\begin{aligned} p_r &= \frac{\partial \mathcal{L}}{\partial \dot{r}} \\ &= m\dot{r} \\ p_\phi &= \frac{\partial \mathcal{L}}{\partial \dot{\phi}} \\ &= mr^2 \dot{\phi} \end{aligned}$$

Note that once again, p_ϕ is the angular momentum (see the Lagrangian treatment of the central force problem in Section 2.4.4). We can now write down the Hamiltonian:

$$\begin{aligned} \mathcal{H} &= m\dot{r}^2 + mr^2\dot{\phi}^2 - \frac{1}{2} m \left(\dot{r}^2 + r^2 \dot{\phi}^2\right) + U(r) \\ &= \frac{1}{2} m\dot{r}^2 + \frac{1}{2} mr^2 \dot{\phi}^2 + U(r) \\ &= \frac{p_r^2}{2m} + \frac{p_\phi^2}{2mr^2} + U(r) \end{aligned}$$

Unsurprisingly, since $\mathcal{L}$ has no explicit time dependence, the Hamiltonian is just the sum of kinetic and potential energy.

We can now apply Hamilton's equations to r:

$$\begin{aligned}\frac{\partial \mathcal{H}}{\partial r} &= -\dot{p}_r \\ -\frac{p_\phi^2}{mr^3} + \frac{dU}{dr} &= -\dot{p}_r \\ \frac{\partial \mathcal{H}}{\partial p_r} &= \dot{r} \\ \frac{p_r}{m} &= \dot{r}\end{aligned}$$

Now we can write down the set of the equations for ϕ:

$$\begin{aligned}\frac{\partial \mathcal{H}}{\partial \phi} &= -\dot{p}_\phi \\ 0 &= -\dot{p}_\phi \\ \frac{\partial \mathcal{H}}{\partial p_\phi} &= \dot{\phi} \\ \frac{p_\phi}{mr^2} &= \dot{\phi}\end{aligned}$$

We see that two of the four equations just give us the definitions of the ps in terms of the coordinates, while the other two carry the dynamical information about the system. The fact that $\dot{p}_\phi = 0$ tells us that angular momentum is conserved. We can look at the first equation we wrote down:

$$-\frac{p_\phi^2}{mr^3} + U'(r) = -m\ddot{r}$$

We can eliminate p_ϕ:

$$m\ddot{r} = -\frac{\partial U}{\partial r} + \frac{l^2}{mr^3}$$

This agrees with our Lagrangian result for the central force problem.

2.6.2 Phase Space Orbits

We can rewrite Hamilton's equations as:

$$\begin{aligned}\dot{q}_i &= \frac{\partial \mathcal{H}}{\partial p_i} \\ &= f_i(q_j, p_j) \\ \dot{p}_i &= -\frac{\partial \mathcal{H}}{\partial q_i} \\ &= g_i(q_j, p_j)\end{aligned}$$

We now introduce the $2n$ dimensional vectors:

$$\boldsymbol{z} = \begin{pmatrix} q_1 \\ \vdots \\ q_n \\ p_1 \\ \vdots \\ p_n \end{pmatrix} \qquad \boldsymbol{h} = \begin{pmatrix} f_1(q_j, p_j) \\ \vdots \\ f_n(q_j, p_j) \\ g_1(q_j, p_j) \\ \vdots \\ g_n(q_j, p_j) \end{pmatrix}$$

We can then rewrite Hamilton's equations as a vector equation:

$$\dot{\boldsymbol{z}} = \boldsymbol{h}(\boldsymbol{z})$$

$\boldsymbol{z}$ defines the "position" of the system in phase space. This first order differential equation tells us how the system is evolving in phase space. Any point $\boldsymbol{z}_0$ defines a possible initial condition of the system. Hamilton's equations then define a unique "phase space orbit" or trajectory $\boldsymbol{z}(t)$ which states from $\boldsymbol{z}_0$ at time $t = t_0$.

One thing that is clear from this is that there is only a single orbit that passes through each point in phase space, provided that the Lagrangian has no explicit time dependence.

Let us do an example. Consider the one-dimensional harmonic oscillator:

$$\mathcal{L} = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$$

We can compute the momentum:

$$\begin{aligned} p &= \frac{\partial \mathcal{L}}{\partial \dot{x}} \\ &= m\dot{x} \end{aligned}$$

Eliminating $\dot{x}$ in favor of p, and writing out the Hamiltonian:

$$\begin{aligned} \mathcal{H} &= p\dot{x} - \mathcal{L} \\ &= \frac{p^2}{2m} + \frac{1}{2}kx^2 \end{aligned}$$

As expected, since the Lagrangian has no explicit time dependence, we recover the energy in the system.

We can now apply Hamilton's equations:

$$\begin{aligned} \frac{\partial \mathcal{H}}{\partial p} &= \dot{x} \\ \frac{p}{m} &= \dot{x} \\ \frac{\partial \mathcal{H}}{\partial x} &= -\dot{p} \\ kx &= -\dot{p} \end{aligned}$$

Eliminating p, we have that

$$kx = -m\ddot{x}$$

as expected, and by convention, we defined $\omega^2 = \frac{k}{m}$:

$$\ddot{x} = -\omega^2 x$$

Which has solutions

$$x = A\cos(\omega t - \delta)$$
$$p = -mA\omega\sin(\omega t - \delta)$$

Let us now trace out the trajectory in phase space that this oscillator takes.

The phase space for this system is 2 dimensional, with coordinates given by (x, p). Rewriting the expressions for the coordinates:

$$\frac{x}{A} = \cos(\omega t - \delta)$$
$$\frac{p}{mA\omega} = -\sin(\omega t - \delta)$$

Using the fact that $\cos^2\theta + \sin^2\theta = 1$, we have that

$$\frac{x^2}{A^2} + \frac{p^2}{m^2A^2\omega^2} = 1$$

This is the equation that defines an ellipse. The phase space orbit looks like an ellipse. Note that this is going clockwise, if we're sitting on the x axis, the $\sin(\omega t - \delta)$ term will be positive for small t, and thus p will become negative, so we have to be going clockwise.

2.6.3 Conservation Laws in the Hamiltonian Formalism

We first show that if $\mathcal{H}$ does not depend explicitly on time, $\mathcal{H} = \mathcal{H}(p, q)$, then $\mathcal{H}$ is a constant of the motion. We can compute $\frac{d\mathcal{H}}{dt}$:

$$\frac{d\mathcal{H}}{dt} = \sum_i \left[\frac{\partial\mathcal{H}}{\partial q_i}\dot{q}_i + \frac{\partial\mathcal{H}}{\partial p_i}\dot{p}_i\right]$$

From Hamilton's equations, $\frac{\partial\mathcal{H}}{\partial q_i} = -\dot{p}_i$ and $\frac{\partial\mathcal{H}}{\partial p_i} = \dot{q}_i$. Inserting these:

$$\frac{d\mathcal{H}}{dt} = \sum_i \left[(-\dot{p}_i)\,\dot{q}_i + (\dot{q}_i)\,\dot{p}_i\right]$$
$$= 0$$

If there was explicit time dependence, we would have an additional $\frac{\partial\mathcal{H}}{\partial t}$ term. Thus, in the case of no explicit time dependence, $\mathcal{H}$ is a constant.

Next, let us show that if q_i is a cyclic coordinate (it does not appear in the Lagrangian), then its conjugate momentum p_i is conserved. This can easily be shown by noting that

$$-\dot{p}_i = \frac{\partial\mathcal{H}}{\partial q_i}$$

And if $\dot{q}_i$ does not appear in the Lagrangian, then $\mathcal{H}$ will also not contain q_i. Thus this derivative is 0, and $\dot{p}_i = 0$, so p_i is conserved.

2.6.4 Hamilton's Equations from a Least Action Principle

We have extremized the action in order to obtain the Euler-Lagrange equations:

$$S = \int_{t_1}^{t_2} dt\, \mathcal{L}\left(q_i, \dot{q}_i, t\right)$$

Where $\delta q\left(t_1\right) = \delta q\left(t_2\right) = 0$. What we now want to do is to obtain Hamilton's equations from the least action principle.

We define the action in the same way:

$$\begin{aligned} S &= \int_{t_1}^{t_2} dt\, \mathcal{L}\left(q_i, \dot{q}_i, t\right) \\ &= \int_{t_1}^{t_2} dt\, \left(\sum_i p_i \dot{q}_i - \mathcal{H}\right) \end{aligned}$$

Let us treat S as a functional of $q_i\left(t\right)$ and $p_i\left(t\right)$, and vary it with respect to these functions independently. This is different than the Lagrangian case, where we considered variations in only the path $q_i\left(t\right)$. In that case, $\dot{q}_i\left(t\right)$ does not vary independently. Instead, we will treat q and p on the same footing, as we do when we define the Hamiltonian (as a function of both q and p).

We can compute the change in the action:

$$\delta S = \int_{t_1}^{t_2} dt\, \left[\sum_i \delta p_i \dot{q}_i + p_i \delta \dot{q}_i - \sum_i \frac{\partial \mathcal{H}}{\partial p_i} \delta p_i + \frac{\partial \mathcal{H}}{\partial q_i} \delta q_i\right]$$

Now we integrate by parts, $p_i \delta \dot{q}_i = \frac{d}{dt}\left(p_i \delta q_i\right) - \delta q_i \dot{p}_i$, and then gather terms:

$$\delta S = \int_{t_1}^{t_2} dt\, \left[\sum_i \left(\dot{q}_i - \frac{\partial \mathcal{H}}{\partial p_i}\right) \delta p_i - \sum_i \left(\dot{p}_i + \frac{\partial \mathcal{H}}{\partial q_i}\right) \delta q_i + \sum_i \frac{d}{dt}\left(p_i \delta q_i\right)\right]$$

Now noting that the integral of the time derivative term can be computed using the FTC:

$$\int dt\, \frac{d}{dt}\left(\sum_i p_i \delta q_i\right) = \sum_i p_i \delta q_i|_{t_2} - p_i \delta q_i|_{t_1}$$

Since the variation at the endpoints is 0, this term is zero, and drops out of the integral. Thus we are left with

$$\delta S = \int_{t_1}^{t_2} dt\, \left[\sum_i \left(\dot{q}_i - \frac{\partial \mathcal{H}}{\partial p_i}\right) \delta p_i - \sum_i \left(\dot{p}_i + \frac{\partial \mathcal{H}}{\partial q_i}\right) \delta q_i\right]$$

Thus we have that these two terms must vanish at the extremum:

$$\begin{aligned} \dot{q}_i &= \frac{\partial \mathcal{H}}{\partial p_i} \\ \dot{p}_i &= -\frac{\partial \mathcal{H}}{\partial q_i} \end{aligned}$$

Which is Hamilton's equations.

There is one interesting feature of this derivation, which is that we exploit the fact that $\delta q_i = 0$ at the endpoints, but we never need the condition that $\delta p_i = 0$ at the endpoints, so we break the "symmetry" between q and p[3].

[3] In the path integral formulation of quantum mechanics, we implicitly break this symmetry of holding x and p fixed

2.6.5 Liouville's Theorem

Theorem 2.2. ***Liouville's Theorem.*** *Consider a region in phase space, and follow its evolution over time. In general, the shape of the region will change. The volume of this region in phase space remains constant with respect to time.*

Proof. Let us consider the 1 dimensional case (the general case follows from the 1D case). The idea of the proof is to consider a region, which is composed of an infinite number of tiny rectangles in phase space, and we will show that the area of each tiny rectangle remains constant, even though the rectangles change in shape. Therefore the overall area of any region must also remain constant while it changes shape.

Consider a rectangular element given by the points (q_A, p_A), (q_B, p_B), (q_C, p_C), and (q_D, p_D). These, after some time, are shifted to (q'_A, p'_A), (q'_B, p'_B), (q'_C, p'_C), and (q'_D, p'_D), respectively. The argument for why this region remains closed is that the trajectory for any point in phase space is unique, and the boundary points will remain the boundary points, and points inside the region will never leave the region.

Since the original shape was chosen to be a rectangle, we have that $qA = q_B$, $p_A = p_B$, $q_B = q_C$, and $p_C = p_B$. We define $q_B - q_A = q_C - q_D = \Delta q$, and $p_D - p_A = p_C - p_B = \Delta p$. Consider the evolution of our region for an infinitesimal timestep Δt. Note that Δq, Δp, and Δt are all infinitesimal, but not comparably infinitesimal:

$$\dot{q}\Delta t \ll \Delta q$$
$$\dot{p}\Delta t \ll \Delta p$$

The distance travelled in our timestep is much less than the length of the rectangle along the q axis.

Now applying Hamilton's equations:

$$\dot{p} = -\frac{\partial \mathcal{H}}{\partial q}$$
$$\dot{q} = \frac{\partial \mathcal{H}}{\partial p}$$

For simplicity, let us assume that $\mathcal{H}$ has no explicit time dependence.

We can compute q'_A :

$$q'_A = q_A + \int_{t_0}^{t_0+\Delta t} \frac{\partial \mathcal{H}}{\partial p}\, dt$$

We now Taylor expand $\frac{\partial \mathcal{H}}{\partial p}$:

$$\frac{\partial \mathcal{H}}{\partial p} = \frac{\partial \mathcal{H}}{\partial p}\Big|_{p_A, q_A} + \left[\left(\frac{\partial^2 \mathcal{H}}{\partial p^2}\right)_{q_A, p_A}\left(\frac{dp}{dt}\right)_{q_A, p_A}(t - t_0) + \left(\frac{\partial^2 \mathcal{H}}{\partial p \partial q}\right)_{q_A, p_A}\left(\frac{dq}{dt}\right)_{q_A, p_A}(t - t_0)\right]$$

due to the uncertainty principle, if we hold $\Delta x = 0$ at the endpoints, p at the endpoints must necessarily be allowed to float.

If we insert this into the integral that we have, we see that the second term will give us terms of order $(\Delta t)^2$, which we neglect. For this reason, we only keep the first term:

$$\begin{aligned} q'_A &= q_A + \int_{t_0}^{t_0+\Delta t} \left(\frac{\partial \mathcal{H}}{\partial p}\right)_{q_A,p_A} dt \\ &= q_A + \left(\frac{\partial \mathcal{H}}{\partial p}\right)_{q_A,p_A} \Delta t \end{aligned}$$

We can repeat this process for q'_B:

$$\begin{aligned} q'_B &= q_B + \left(\frac{\partial \mathcal{H}}{\partial p}\right)_{q_B,p_B} \Delta t \\ &= q_A + \Delta q + \left(\frac{\partial \mathcal{H}}{\partial p}\right)_{q_A+\Delta q,p_A} \Delta t \\ &= q_A + \Delta q + \left(\frac{\partial \mathcal{H}}{\partial p}\right)_{q_A,p_A} \Delta t + \left(\frac{\partial^2 \mathcal{H}}{\partial q \partial p}\right)_{q_A,p_A} \Delta t \Delta q \end{aligned}$$

We see that q'_B and q'_A have a relative change, given by only the term

$$\left(\frac{\partial^2 \mathcal{H}}{\partial q \partial p}\right)_{q_A,p_A} \Delta t \Delta q$$

We can repeat this process for all 8 shifted coordinates, and then compute the area of the new region. The rectangle becomes a parallelogram, and we know the relative locations of the vertices, and therefore we can compute the area, by taking the cross product of the vectors representing two adjacent sides. We find that the area is given by

$$A = \Delta p \Delta q + \mathcal{O}\left(\Delta p \Delta q \Delta t^2\right)$$

We recover the original area, and we pick up a negligible term, and thus the area of the infinitesimal rectangle is conserved. From this, the area of any region is conserved. □

2.6.6 Liouville's Equation

Consider a scenario in which we have a very large number N of identical, noninteracting systems, all governed by the same Hamiltonian. We can define a "density function" which gives information about the number of systems in the neighborhood of a point (p, q) in phase space. This is an important consideration when doing chaos theory, plasma physics, and accelerator physics. The density function $\rho(q, p, t)$ is integrated over an area in phase space:

$$\int_{q,p}^{q+\Delta q,p+\Delta p} \rho(q,p,t) \prod_i dq_i dp_i = n$$

Where n is the number of systems in the region of phase space we are integrating over. Integrating over all space, we get the total number of systems:

$$\int \rho(q,p,t) \prod_i dq_i dp_i = N$$

Let us consider a tiny region of phase space, and at some time there is n systems in this region. The region evolves in time according to Liouville's theorem, maintaining constant volume. We also claim that n is conserved, because for a system to leave, its trajectory would cross the trajectory of a boundary point for the region, which is disallowed. Thus, as time evolves, $\frac{dn}{dt} = 0$. If we consider an infinitesimally small region,

$$\frac{d}{dt}\left[\rho\left(q,p,t\right)\Delta V\right] = 0$$

We know that $\frac{d}{dt}\Delta V = 0$, by Liouville's theorem (2.2), and thus

$$\frac{d}{dt}\left(\rho\left(q,p,t\right)\right) = 0$$

That is, the density of systems in a region of phase space remains constant as we evolve in time.

We can also write out the time derivative:

$$\frac{d\rho}{dt} = \sum_i \frac{\partial\rho}{\partial p_i}\frac{dp_i}{dt} + \sum_i \frac{\partial\rho}{\partial q_i}\frac{dq_i}{dt} + \frac{\partial\rho}{\partial t}$$

Applying Hamilton's equations to the time derivatives of p_i and q_i, and setting this to zero:

$$\left(\frac{\partial\rho}{\partial t}\right)_{p_i,q_i} = \sum_i \left(\frac{\partial\rho}{\partial p_i}\frac{\partial\mathcal{H}}{\partial q_i} - \frac{\partial\rho}{\partial q_i}\frac{\partial\mathcal{H}}{\partial p_i}\right) \tag{7}$$

This is Liouville's equation. This allows us to look at some static region of phase space, and determine the flux of the number of systems in the region over time. The previous equation had us tracking the evolution of a region, and we found that the number of systems inside the region was constant, while this equation keeps the region fixed and tracks the evolution of the number of systems entering and leaving the system.

2.6.7 Poincaré Recurrence Theorem

Poincaré's recurrence theorem applies to systems in which the phase space is bounded (it has finite volume). For example, if we have a bound on the total energy, then we cannot move arbitrarily far from the starting point in phase space.

Theorem 2.3. ***Poincaré Recurrence Theorem.*** *Consider an initial point P in phase space. Then for any neighborhood D_0 of P, there exists a point $P' \in D_0$ which will return to D_0 in finite time.*

In theory, if we lived in a closed universe governed by classical mechanics, this theorem would imply that at some point you would be taking this course again. Obviously, our universe isn't governed by classical mechanics, and it is not clear that it is a closed system so this is (hopefully) not true.

Proof. We have some region D_0, which after time T, becomes the region D_1. By Liouville's theorem (2.2), the volumes of these two regions must be equivalent. Let D_k be the region after time kT, where $k \in \mathbb{Z}$. There must exist $k, k' \in \mathbb{Z}$ such that some regions of D_k and $D_{k'}$ overlap,

$$D_k \cap D_{k'} \neq \varnothing$$

This is because our phase space is bounded, and so at some point the regions must start to overlap, it can only fill up the space for some finite number of times, eventually we must have some overlap with the region for some previous time. This is why the condition of the phase space being bounded is required, this is not true for an unbounded phase space. In the unbounded case:

$$\bigcup_{k=0}^{k=\infty} D_k = \infty$$

Note that we have not proved the theorem already, we need to show that for any k, there is some k' for which there is overlap.

Since the evolution of regions in phase space is fixed, if regions D_k and $D_{k'}$ have some overlap, that set of overlapping points will continue to overlap throughout time, for D_{k+1} and $D_{k'+1}$, etc. However, not only is the motion forwards in time fixed, but the motion backwards in time is also fixed. This means that they also have an overlapping region for D_{k-1} and $D_{k'-1}$, and for all D_{k-a} and $D_{k'-a}$, where $a \in \mathbb{Z}$. This means that at some point we can have $D_{k-a} = D_0$, which must overlap with $D_{k'-k}$:

$$D_0 \cap D_{k'-k} \neq \varnothing$$

Again, this results from the fact that the mapping $D_k \to D_{k+1}$ is invertible, we can track backwards as well as forwards in time.

□

2.6.8 Poisson Brackets

Let $f(\boldsymbol{q}, \boldsymbol{p})$ and $g(\boldsymbol{q}, \boldsymbol{p})$ be two functions in phase space. The Poisson bracket is defined as

$$\{f, g\} = \sum_i \left(\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial q_i} \frac{\partial f}{\partial p_i} \right)$$

Let us consider the properties of Poisson brackets. The first is that the exchange of arguments picks up a negative:

$$\{f, g\} = -\{g, f\}$$

We also have a linearity property:

$$\{\alpha f + \beta g, h\} = \alpha\{f, h\} + \beta\{g, h\}$$

Where α and β are constants.

We also have a relation that is reminiscent of commutator properties from QM:

$$\{fg, h\} = f\{g, h\} + \{f, h\}g$$
$$\{h, fg\} = f\{h, g\} + \{h, f\}g$$

We also have what is known as the Jacobi identity:

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$$

All of these follow from the definition of the Poisson bracket. Additionally, all of the previous identities are also true of matrix commutators, $\left[\hat{f}, \hat{g}\right]$.

The close relationship to QM is further evidenced by Poisson bracket relations that are similar to the canonical commutation relations:

$$\begin{aligned}\{q_i, q_j\} &= 0\\ \{p_i, p_j\} &= 0\\ \{q_i, p_j\} &= \delta_{ij}\end{aligned}$$

For any function $f(q, p, t)$, we have that

$$\frac{df}{dt} = \{f, \mathcal{H}\} + \frac{\partial f}{\partial t}$$

This is essentially how operators evolve in the Heisenberg picture.

Proof. The proof of this relation is simple, we can write the total time derivative of f:

$$\frac{df}{dt} = \sum_i \left(\frac{\partial f}{\partial q_i}\dot{q}_i + \frac{\partial f}{\partial p_i}\dot{p}_i\right) + \frac{\partial f}{\partial t}$$

Now applying Hamilton's equations, we have that $\dot{q}_i = \frac{\partial \mathcal{H}}{\partial p_i}$, and $\dot{p}_i = -\frac{\partial \mathcal{H}}{\partial q_i}$:

$$\begin{aligned}\frac{df}{dt} &= \sum_i \left(\frac{\partial f}{\partial q_i}\frac{\partial \mathcal{H}}{\partial p_i} - \frac{\partial g}{\partial p_i}\frac{\partial \mathcal{H}}{\partial q_i}\right) + \frac{\partial f}{\partial t}\\ &= \{f, \mathcal{H}\} + \frac{\partial f}{\partial t}\end{aligned}$$

□

From this, any function $I(p, q)$ that does not depend explicitly on time, and "Poisson commutes" with the Hamiltonian:

$$\{I, \mathcal{H}\} = 0$$

must necessarily be a constant of the motion, since $\frac{dI}{dt} = \{I, \mathcal{H}\}$.

As an example, consider the case where q_i is a cyclic coordinate. In this case, the Lagrangian only explicitly depends on the derivative of q_i, not on q_i itself. In this case, $\mathcal{H}$ will not add explicit dependence on q_i, which we argued earlier. In this case,

$$\begin{aligned}\dot{p}_i &= \{p_i, \mathcal{H}\}\\ &= \sum_j \left(\frac{\partial p_i}{\partial q_j}\frac{\partial \mathcal{H}}{\partial p_j} - \frac{\partial \mathcal{H}}{\partial q_j}\frac{\partial p_i}{\partial p_j}\right)\end{aligned}$$

We see that $\frac{\partial p_i}{\partial q_j} = 0$, and $\frac{\partial p_i}{\partial p_j} = \delta_{ij}$, so the whole thing becomes:

$$\dot{p}_i = -\frac{\partial \mathcal{H}}{\partial q_i}$$

Which, since q_i is cyclic, must be zero. Thus $\dot{p}_i = 0$ for a cyclic coordinate, and p_i is constant. This matches what we expect, the conjugate momentum of a cyclic coordinate is 0.

If I and J are constants of the motion, and therefore $\{I, \mathcal{H}\} = \{J, \mathcal{H}\} = 0$, their Poisson commutator $\{I, J\}$ must also be a constant of the motion:

$$\{\{I, J\}, \mathcal{H}\} = 0$$

This follows from the Jacobi identity:

$$\{\{I, J\}, \mathcal{H}\} + \{\{J, \mathcal{H}\}, I\} + \{\{\mathcal{H}, I\}, J\} = 0$$

Since the second and third terms must be zero, the first term must be zero.

Let's do an example. Consider $\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p}$. In components:

$$\begin{aligned} L_1 &= x_2 p_3 - x_3 p_2 \\ L_2 &= x_3 p_1 - x_1 p_3 \\ L_3 &= x_1 p_2 - x_2 p_1 \end{aligned}$$

Now let us consider the Poisson bracket $\{L_1, L_2\}$

$$\begin{aligned} \{L_1, L_2\} &= \{x_2 p_3 - x_3 p_2, x_3 p_1 - x_1 p_3\} \\ &= \{x_2 p_3, x_3 p_1\} + \{x_3 p_2, x_1 p_3\} \\ &= x_2 \{p_3, x_3\} p_1 + x_1 \{x_3, p_3\} p_2 \\ &= -x_2 p_1 + x_1 p_2 \\ &= x_1 p_2 - x_2 p_1 \\ &= L_3 \end{aligned}$$

We see that we recover the same commutator as in QM, the Poisson commutator of L_1 and L_2 gives us L_3.

Similarly, one can show that $\{L^2, L_i\} = 0$, where $L^2 = \sum_i L_i^2$. This once again mimics what we see in quantum mechanics.

2.6.9 Canonical Transformations

In the Hamiltonian formalism, q_i and p_i are on equal footing, which allows for a much greater class of coordinate transformations:

$$\begin{aligned} q_i &\to Q_i(q, p, t) \\ p_i &\to P_i(q, p, t) \end{aligned}$$

However, not all transformations of this form are allowed, we have the restriction that the new coordinates should evolve in time as governed by Hamilton's equations. This means that we transform the Hamiltonian:

$$\mathcal{H}(q, p, t) \to \mathcal{K}(Q, P, t)$$

Such that

$$\frac{\partial \mathcal{K}}{\partial \dot{P}_i} = \dot{Q}_i$$
$$\frac{\partial \mathcal{K}}{\partial \dot{Q}_i} = -\dot{P}_i$$

Transformations that satisfy this condition are known as **canonical transformations**.

We will focus on *restricted* canonical transformations, which are those that are not time dependent:

$$q_i \to Q\left(q,p\right)$$
$$p_i \to P\left(q,p\right)$$
$$\mathcal{H}\left(q,p\right) = \mathcal{K}\left(Q,P\right)$$

What are the operations that we can actually do? We first write Hamilton's equations in a symmetric form. We define the vector $\boldsymbol{z} = (q_1, q_2, \ldots, q_n, p_1, p_2, \ldots, p_n)$. This is a $2n$ dimensional vector. We also define a $2n \times 2n$ matrix $\hat{J}$, which we define as a block matrix

$$\hat{J} = \begin{bmatrix} 0_{n\times n} & \mathbb{I}_{n\times n} \\ -\mathbb{I}_{n\times n} & 0_{n\times n} \end{bmatrix}$$

This is the invariant metric for symplectic transformations, and it shows up a lot in physics. In this notation, we can write Hamilton's equations as:

$$\dot{\boldsymbol{z}} = \hat{J}\frac{\partial \mathcal{H}}{\partial \boldsymbol{z}}$$

The derivative $\partial\mathcal{H}/\partial\boldsymbol{z}$ is the gradient in the space of $\boldsymbol{z}$ vectors. In component form, we recover the correct equations:

$$\dot{z}_i = \sum_j \hat{J}_{ij}\frac{\partial \mathcal{H}}{\partial z_j}$$

Now let us make the coordinate transformation from q and p to Q and P. This means that

$$z_i \to w_i\left(\boldsymbol{z}\right)$$

Now let us figure out what the conditions are for this transformation from z_i to w_i. We start by computing $\dot{w}_i$:

$$\begin{aligned}\dot{w}_i &= \sum_j \frac{\partial w_i}{\partial z_j}\dot{z}_j \\ &= \sum_j \sum_k \frac{\partial w_i}{\partial z_j}\hat{J}_{jk}\frac{\partial \mathcal{H}}{\partial z_k}\end{aligned}$$

Now we can write $\frac{\partial \mathcal{H}}{\partial z_k} = \sum_l \frac{\partial \mathcal{H}}{\partial w_l}\frac{\partial w_l}{\partial z_k}$:

$$\begin{aligned}\dot{w}_i &= \sum_j \sum_k \sum_l \frac{\partial w_i}{\partial z_j}\hat{J}_{jk}\frac{\partial \mathcal{H}}{\partial w_l}\frac{\partial w_l}{\partial z_k} \\ &= \sum_{j,k,l} \frac{\partial w_i}{\partial z_j}\hat{J}_{jk}\frac{\partial w_l}{\partial z_k}\frac{\partial \mathcal{H}}{\partial w_l}\end{aligned}$$

The thing to notice is that $\frac{\partial w_i}{\partial z_j}$ are the elements of the Jacobian matrix[4] J for the transformation from z to w:

$$J_{ij} = \frac{\partial w_i}{\partial z_j}$$

From this, we can see that we have

$$\dot{w}_i = \sum_{j,k,l} J_{ij}\hat{J}_{jk}J_{lk}\frac{\partial \mathcal{H}}{\partial w_l}$$

From this, we note that we can write this as a vector equation:

$$\dot{\boldsymbol{w}} = \left(J\hat{J}J^T\right)\frac{\partial \mathcal{H}}{\partial \boldsymbol{w}}$$

If the transformation is canonical, then we have that

$$\dot{\boldsymbol{J}} = \hat{J}\frac{\partial \mathcal{H}}{\partial \boldsymbol{w}}$$

So we must have that

$$\boxed{J\hat{J}J^T = \hat{J}}$$

If this condition holds, the Jacobian is said to be *symplectic*. In terms of the coordinates, this means that

$$\sum_{j,k} \frac{\partial w_i}{\partial z_j}\hat{J}_{jk}\frac{\partial w_l}{\partial z_k} = \hat{J}_{il}$$

From this, we see some of the group structure of the transformations start to show, for example, doing two canonical transformations in a row must be a canonical transformation.

Now let us show that this condition is equivalent to the requirement that the new coordinates Q_i and P_i satisfy the conditions:

$$\begin{aligned}\{Q_i, Q_j\} &= 0 \\ \{P_i, P_j\} &= 0 \\ \{Q_i, P_j\} &= \delta_{ij}\end{aligned}$$

[4] $\hat{J}$ and J are different matrices!

Proof. We can look at the Jacobian matrix schematically, as a block matrix:

$$J = \begin{bmatrix} \frac{\partial Q_i}{\partial q_j} & \frac{\partial Q_i}{\partial p_j} \\ \frac{\partial P_i}{\partial q_j} & \frac{\partial P_i}{\partial p_j} \end{bmatrix}$$

We can now look at $J\hat{J}J^T$ in terms of $n \times n$ block matrices:

$$\left(J\hat{J}J^T\right)_{il} \begin{bmatrix} \frac{\partial Q_i}{\partial q_j} & \frac{\partial Q_i}{\partial p_j} \\ \frac{\partial P_i}{\partial q_j} & \frac{\partial P_i}{\partial p_j} \end{bmatrix} \begin{bmatrix} 0 & \delta_{jk} \\ -\delta_{jk} & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial Q_l}{\partial q_k} & \frac{\partial P_l}{\partial q_k} \\ \frac{\partial Q_l}{\partial p_k} & \frac{\partial P_l}{\partial p_k} \end{bmatrix}$$

Now doing this matrix multiplication out, we are left with

$$\left(J\hat{J}J^T\right)_{il} = \begin{bmatrix} \{Q_i, Q_l\} & \{Q_i, P_l\} \\ \{P_i, Q_l\} & \{P_i, P_l\} \end{bmatrix}$$

Fro this to be canonical, we need this to be equal to $\hat{J}_{il}$:

$$\begin{bmatrix} \{Q_i, Q_l\} & \{Q_i, P_l\} \\ \{P_i, Q_l\} & \{P_i, P_l\} \end{bmatrix} = \begin{bmatrix} 0 & \delta_{il} \\ -\delta_{il} & 0 \end{bmatrix}$$

From this, we can read off the condition for the transformation to be canonical:

$$\begin{aligned} \{Q_i, Q_j\} &= 0 \\ \{P_i, P_j\} &= 0 \\ \{Q_i, P_j\} &= \delta_{ij} \end{aligned}$$

Which is exactly what we expect. □

Now let us pivot and consider the Poisson brackets once more.

Theorem 2.4. *The Poisson bracket is invariant under canonical transformations. Under $q \to Q$ and $p \to P$, the Poisson bracket $\{f, g\}$ remains the same:*

$$\begin{aligned} \{f, g\} &= \sum_i \left(\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial q_i} \frac{\partial f}{\partial p_i} \right) \\ &= \sum_i \left(\frac{\partial f}{\partial Q_i} \frac{\partial g}{\partial P_i} - \frac{\partial g}{\partial q_i} \frac{\partial f}{\partial P_i} \right) \end{aligned}$$

Proof. We first note that we can rewrite the Poisson bracket:

$$\begin{aligned} \{f, g\} &= \sum_i \left(\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial q_i} \frac{\partial f}{\partial p_i} \right) \\ &= \sum_{i,j} \frac{\partial f}{\partial z_i} \hat{J}_{ij} \frac{\partial g}{\partial z_j} \end{aligned}$$

Under a change of coordinates from $z \to w(z)$, then we can rewrite the derivatives:

$$\frac{\partial f}{\partial z_i} = \sum_k \frac{\partial f}{\partial w_k} \underbrace{\frac{\partial w_k}{\partial z_i}}_{J_{ki}}$$

Where J is the Jacobian. We can then rewrite the Poisson bracket:

$$\begin{aligned}
\{f, g\} &= \sum_i \left(\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial q_i} \frac{\partial f}{\partial p_i} \right) \\
&= \sum_{i,j} \frac{\partial f}{\partial z_i} \hat{J}_{ij} \frac{\partial g}{\partial z_j} \\
&= \sum_{i,j,k,l} \frac{\partial f}{\partial w_k} \underbrace{J_{ki} \hat{J}_{ij} J_{lj}}_{(J \hat{J} J^T)_{kl}} \frac{\partial g}{\partial w_l}
\end{aligned}$$

Now noting that by definition, a canonical transformation has $J \hat{J} J^T = \hat{J}$, we are left with

$$\begin{aligned}
\{f, g\} &= \sum_{kl} \left(\frac{\partial f}{\partial w_k} \right) \hat{J}_{kl} \frac{\partial g}{\partial w_l} \\
&= \sum_i \left(\frac{\partial f}{\partial Q_i} \frac{\partial g}{\partial P_i} - \frac{\partial g}{\partial Q_i} \frac{\partial f}{\partial P_i} \right)
\end{aligned}$$

Thus for any canonical transformation, the Poisson brackets remain invariant. □

2.6.10 Action-Angle Variables

Consider a Hamiltonian $\mathcal{H}(q, p)$, with no explicit time dependence, restricted to a single degree of freedom. We now make the assumption that q is bounded, $q \in [q_1, q_2]$. Because of these assumptions, by the Poincaré recurrence theorem, the system will undergo periodic motion. However, we can make a canonical transformation from (p, q) to (I, θ), such that θ is cyclic, $\mathcal{H}$ has no explicit θ dependence, $\mathcal{H}(I)$. For this 1D system, it is *always* possible to make such a transformation.

Hamilton's equations then tell us that

$$\begin{aligned}
\frac{\partial \mathcal{H}}{\partial \theta} &= \dot{I} \\
&= 0 \\
\frac{\partial \mathcal{H}}{\partial I} &= \dot{\theta}
\end{aligned}$$

The first equation tells us that I is a constant, and since $\dot{\theta}$ is a function of only I, $\dot{\theta}$ must also be constant. By convention, we normalize I and θ, such that $\dot{\theta} = \omega$, which is the angular frequency of oscillations. The momentum I is known as the "action variable", and the cyclic coordinate θ is known as the "angle variable".

Let us determine the action-angle variables for a 1D system where:

$$\mathcal{H} = \frac{p^2}{2m} + V(q)$$

Where we also bound our system so that q is between q_1 and q_2.

Since I is a constant of the motion, it must be some function of the total energy of the system. This is because $\mathcal{H}$ must be a function of only I, and $\mathcal{H}$ is the total energy in the system. Another reasoning is that we can't introduce another constant of the motion other than the energy, so I has to be related to E in some way.

We can write out $\dot{\theta}$:

$$\begin{aligned}\dot{\theta} &= \frac{\partial \mathcal{H}}{\partial I} \\ &= \frac{d\mathcal{H}}{dI} \\ &= \frac{dE}{dI} \\ &= \omega\end{aligned}$$

Where we have used the fact that $\mathcal{H}$ is independent of θ, then that $\mathcal{H} = E$, and then the definition of $\dot{\theta}$ in terms of the angular frequency. We can't just integrate this because ω may be a function of I. Now we cheat a bit, and we claim the correct answer, and then we will verify it. We claim that the correct choice of I is

$$\boxed{I = \frac{1}{2\pi} \oint p \, dq}$$

This integral is the area enclosed by the orbit in phase space, scaled by 2π. This is reminiscent of the Bohr-Sommerfeld quantization from QM.

Now let us prove that this choice of I satisfies the action-angle conditions.

Proof. The orbit in phase space is some region that must have symmetry when reflected across the q axis, but not necessarily across the p axis. The region is also bounded so that $q \in [q_1, q_2]$. The area is a function of the energy in the system. The claim is that $\frac{dE}{dI} = \omega$. We need to show that

$$\frac{d}{dE}\left(\oint p \, dq\right) = \frac{2\pi}{\omega}$$

Recall that $p = \sqrt{2m\left(E - V\left(q\right)\right)}$ for this system. As we change E, we need to see how the integral change. There are two effects to changing E. The first is the integrand changes for all q, since p depends on E :

$$p \to p + \left(\frac{\partial p}{\partial E}\right)_q \Delta E$$

The second effect is that the endpoints q_1 and q_2 are altered, since the system can oscillate between different points depending on what the energy is.

We have to account for both of these effects.

Its convenient to look at these separately. First, let us consider the change in the value of the integrand.

We have to compute the integral of the shifted part of p:

$$\oint \left(\frac{\partial p}{\partial E}\right)_q dq = \sqrt{\frac{m}{2}} \oint \frac{dq}{\sqrt{E - V(q)}}$$

Now we note that $\frac{1}{2}m\dot{q}^2 + V(q) = E$, since $\dot{q} = \frac{\partial \mathcal{H}}{\partial p} = \frac{p}{m}$. Using this, we can solve this for $\dot{q}$, and then separate the derivative and integrate both sides:

$$\int dt = \int dq \sqrt{\frac{m}{2}} \frac{1}{\sqrt{E - V(q)}}$$

We see that this is very similar to what we have for the integral that we are trying to compute. This gets us that

$$\oint \left(\frac{\partial p}{\partial E}\right)_q dq = T$$

Where T is the period of the orbit. By definition, $T = \frac{2\pi}{\omega}$. This is the outcome of the first effect, and so we hope that the second effect vanishes.

By inspection, we know that the value of p is very small near the turning points of the system. Thus, when we shift the turning point slightly, we have a negligible contribution to the integral. We can show this (in a hand-wavy way). Setting $V(q_i + \Delta q_i) = E + \Delta E$, we have

$$V'(q_i)\,\Delta q_i = \Delta E$$

Then contributions to $\oint p\,dq$ from the region between q_i and $q_i + \Delta q_i$ is given by

$$\int_{q_i}^{q_i + \Delta q_i} dq \sqrt{2m(E - V(q))} \approx \left(2m \underbrace{(E - V(q_i))}_{\approx 0}\right)^{1/2} \underbrace{\Delta q_i}_{\frac{\Delta E}{V'(q_i)}} = 0$$

Where we note that the difference between the total energy and the potential near the turning points is very close to zero. From this, we see that the second effect is negligible[5], and thus

$$\begin{aligned}\frac{d}{dE}\oint p\,dq &= \oint \frac{\partial p}{\partial E}\,dq \\ &= \frac{2\pi}{\omega}\end{aligned}$$

As expected.

□

We have an expression for the action variable, but what about the angle variable? To obtain the angle variable, we note that

$$t - t_0 = \int_{q(t_0)}^{q(t)} dq \left(\frac{\partial p}{\partial E}\right)_q$$

[5]This is hand-wavy, if we do it out precisely instead of zero, we get something of order $(\Delta q_i)^2$

$$= \frac{d}{dE}\int_{q(t_0)}^{q(t)} p\,dq$$

We can then write $\theta(t)$:

$$\theta(t) = \theta(t_0) + \omega t$$

Setting $\theta(t_0) = 0$ by convention:

$$\begin{aligned}\theta(t) &= \omega t \\ &= \frac{dE}{dI}\frac{d}{dE}\int_{q(t_0)}^{q(t)} p\,dq \\ &= \frac{d}{dI}\int_{q(t_0)}^{q(t)} p\,dq\end{aligned}$$

Now let us consider the flow in phase space. In the usual (p, q) coordinates, we have some closed trajectory. In action-angle coordinates, the orbits are straight lines, with periodic boundary conditions as θ wraps around from 2π to 0. Different values of I are different straight lines.

One thing to keep in mind is that we can determine the period of the orbit T from $\omega = \frac{dE}{dI}$ after calculating $I(E)$.

Let us do an example. Suppose we want to find the action-angle variables for simple harmonic oscillator, and determine the period of oscillation. We have that

$$\begin{aligned}\mathcal{H} &= \frac{p^2}{2m} + \frac{1}{2}kq^2 \\ &= E\end{aligned}$$

From this, we have that $p = \pm\sqrt{2m\left(E - \frac{1}{2}kx^2\right)}$. We can solve for I:

$$\begin{aligned}I &= \frac{1}{2\pi}\oint p\,dx \\ &= \frac{1}{2\pi}\left[2\int_{-A}^{A}\sqrt{2m\left(E - \frac{1}{2}kx^2\right)}\,dx\right]\end{aligned}$$

Where $x = \pm A$ are the turning points, where $E = \frac{1}{2}kA^2$. The factor of 2 is to account for the top and bottom halves of the orbit, which are symmetrical. We can solve this integral:

$$\begin{aligned}I &= \frac{1}{2\pi}\left[2\int_{-A}^{A}\sqrt{2m\left(E - \frac{1}{2}kx^2\right)}\,dx\right] \\ &= \frac{1}{2\pi}\left[2\sqrt{2m}\int_{-A}^{A}\sqrt{\frac{1}{2}kA^2 - \frac{1}{2}kx^2}\,dx\right] \\ &= \frac{1}{2\pi}\left[2\sqrt{2m}\int_{-\pi/2}^{\pi/2}\sqrt{\frac{1}{2}kA^2\cos^2\alpha}A\cos\alpha\,d\alpha\right] \\ &= \frac{1}{2\pi}\left[2\sqrt{km}A^2\int_{-\pi/2}^{\pi/2}\cos^2\alpha\,d\alpha\right]\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2\pi}\left[2\sqrt{km}A^2\int_{-\pi/2}^{\pi/2} d\alpha\, \frac{1}{2} + \frac{1}{2}\cos(2\alpha)\right] \\
&= \frac{1}{2\pi}\left(2\sqrt{km}A^2\frac{\pi}{2}\right) \\
&= \sqrt{\frac{m}{k}}E
\end{aligned}
$$

Where we have used the substitution $x = A\sin\alpha$, and thus $dx = A\cos\alpha$. This tells us that $E = \sqrt{\frac{k}{m}}I$, and thus $\omega = \frac{dE}{dI} = \sqrt{\frac{k}{m}}$, which is what we expect.

Now we want to find the angle variable. We can do this by computing:

$$
\begin{aligned}
\theta &= \frac{d}{dI}\int_{x(t_0)}^{x(t)} p\, dx \\
&= \int_{x(t_0)}^{x(t)} \frac{\partial p}{\partial I}\, dq
\end{aligned}
$$

Now noting that

$$
\begin{aligned}
p &= \sqrt{2m\left(E - \frac{1}{2}kx^2\right)} \\
&= \sqrt{2m\left(\omega I - \frac{1}{2}kx^2\right)}
\end{aligned}
$$

We can then solve for θ:

$$
\begin{aligned}
\theta &= \int^{x(t)} \sqrt{\frac{m}{2}}\frac{\omega\, dx}{\sqrt{\omega I - \frac{1}{2}x^2}} \\
&= \sqrt{\frac{m}{2}}\omega\frac{1}{\sqrt{\frac{1}{2}k}}\int^{x(t)} \frac{dx}{\sqrt{\frac{2\omega}{k}I - x^2}}
\end{aligned}
$$

If we define $\frac{2\omega}{k}I$ as a^2, we are left with an integral of the form

$$
\int \frac{dx}{\sqrt{a^2 - x^2}}
$$

Which can be solved the same way as our previous integral, we get an arcsine:

$$
\begin{aligned}
\theta &= \sqrt{\frac{m}{k}}\omega \arcsin\left(\frac{x}{a}\right) \\
&= \arcsin\left(\frac{xI}{\sqrt{\frac{2\omega}{k}}}\right) \\
&= \arcsin\left(\frac{x}{\sqrt{\frac{2E}{k}}}\right)
\end{aligned}
$$

2.6.11 Integrable Systems

We saw that for 1D systems with a time independent Hamiltonian, we could find action-angle variables, in which one of the variables is cyclic, and the motion in phase space is trivial.

Suppose we now have a system with multiple degrees of freedom. Is it in general possible to do this? We want to make some transformation such that

$$\mathcal{H}(p_i, q_i) \to \mathcal{H}(I_i, \theta_i)$$

Such that $\mathcal{H}(I_i, \theta_i) = \mathcal{H}(I_i)$, all the θ coordinates are cyclic.

The answer is that **no**, this is not possible in general. If such a transformation exists, the system is considered to be **integrable**.

Unfortunately, integrable systems are very rare. Generally speaking, if a classical mechanics problem is solved, it is integrable. In fact, since the harmonic oscillator is integrable, we can map any integrable system to the harmonic oscillator.

We can also discuss Liouville's theorem on integrable systems, which relates the number of constants of the motion to the integrability of the system.

Theorem 2.5. ***Liouville's Theorem on Integrable Systems.*** *In an n-dimensional system, if we can find n independent mutually commuting constants of motion, $(I_1, I_2, \ldots, I_n)$, the system is integrable.*

This tells us that every 1D system with a time independent Hamiltonian is integrable, since the energy is a constant of the motion.

2.6.12 Adiabatic Invariants

Consider a 1D system with a time independent Hamiltonian of the form:

$$\mathcal{H} = \frac{p^2}{2m} + V(q)$$

Let us once again assume that the motion is bounded, $q \in [q_1, q_2]$.

Let the potential depend on some parameter λ, so that $V(q, \lambda)$.

We want to explore what happens as we vary λ very slowly ("adiabatically") with time. We define adiabatic variation to be when $\frac{d\lambda}{dt} \ll \frac{\lambda}{T}$, where T is the period. The change in λ over the timescale of 1 period of the system is negligible.

The first thing to notice is that E is no longer constant, V has λ dependence, which has explicit time dependence. The change in energy over time is given by

$$\begin{aligned}\frac{dE}{dt} &= \left(\frac{\partial \mathcal{H}}{\partial t}\right)_{p,q} \\ &= \left(\frac{\partial \mathcal{H}}{\partial \lambda}\right)_{p,q}\left(\frac{d\lambda}{dt}\right)\end{aligned}$$

We have seen that E is not conserved, but there are specfic combinations of E and λ which remain approximately constant as λ is slowly changed. These combinations are known as **adiabatic invariants**.

For this system, we claim that the adiabatic invariant is

$$I = \frac{1}{2\pi} \oint p\, dq$$

As we shift λ, the area of the orbit in phase space remains constant. The orbits don't strictly close any more, but we can approximate them to be closed to very good approximation, since λ varies very slowly. Now let us show this. In general, the orbit depends on E and λ.

We can consider the change in I:

$$\Delta I = \left(\frac{\partial I}{\partial E}\right)_\lambda \Delta E + \left(\frac{\partial I}{\partial \lambda}\right)_E \Delta\lambda$$
$$\dot{I} = \left(\frac{\partial I}{\partial E}\right)_\lambda \frac{dE}{dt} + \left(\frac{\partial I}{\partial \lambda}\right)_E \frac{d\lambda}{dt}$$

Recall that $\frac{dI}{dE}$ is related to T, $\left(\frac{\partial I}{\partial E}\right)_\lambda = \frac{1}{\omega(\lambda)} = \frac{T(\lambda)}{2\pi}$. We also know that $\frac{dE}{dt}$ can be written as previously derived, in terms of the derivative of the Hamiltonian and $\frac{d\lambda}{dt}$. From this, we have that

$$\dot{I} = \frac{T(\lambda)}{2\pi}\left[\left(\frac{\partial \mathcal{H}}{\partial \lambda}\right)_{p,q} \frac{d\lambda}{dt}\right] + \left(\frac{\partial I}{\partial \lambda}\right)_E \frac{d\lambda}{dt}$$

Now looking at the second term, this is the rate of change of the area with respect to λ, with E held constant.

$$\begin{aligned}\left(\frac{\partial I}{\partial \lambda}\right)_E &= \frac{1}{2\pi}\left(\frac{\partial}{\partial \lambda}\oint p\, dq\right)_E \\ &= \frac{1}{2\pi}\oint \left(\frac{\partial p}{\partial \lambda}\right)_{E,q} dq\end{aligned}$$

Where we have moved the derivative inside the integral by neglecting the contribution of the endpoints. Now we note that we know p in terms of E:

$$p = \sqrt{2m\left(E - V(\lambda, q)\right)}$$

We know that

$$dE = \left(\frac{\partial \mathcal{H}}{\partial q}\right)_{p,\lambda} dq + \left(\frac{\partial \mathcal{H}}{\partial p}\right)_{q,\lambda} dp + \left(\frac{\partial \mathcal{H}}{\partial \lambda}\right)_{p,q} d\lambda$$

For the case where $dE = 0$ and $dq = 0$ (since thats what we want in the derivative of p wrt λ), we have that

$$\left(\frac{\partial \mathcal{H}}{\partial p}\right)_{q,\lambda}\left(\frac{\partial p}{\partial \lambda}\right)_{q,E} + \left(\frac{\partial \mathcal{H}}{\partial \lambda}\right)_{p,q} = 0$$

Going back to $\left(\frac{\partial I}{\partial \lambda}\right)$, we can do a change of variables to a time integral:

$$\left(\frac{\partial I}{\partial \lambda}\right)_E = \frac{1}{2\pi}\oint \left(\frac{\partial p}{\partial \lambda}\right)_{q,E} \frac{dq}{dt}\, dt$$

$$= -\frac{1}{2\pi} \oint \left(\frac{\partial \mathcal{H}}{\partial \lambda}\right)_{p,q} dt$$
$$= -\frac{1}{2\pi} \int_0^{T(\lambda)} \left(\frac{\partial \mathcal{H}}{\partial \lambda}\right)_{p,q} dt$$

Where we have used the fact that $\frac{dq}{dt} = \left(\frac{\partial \mathcal{H}}{\partial p}\right)_{q,\lambda}$, and we have used the relationship derived above to replace the integrand with $\left(\frac{\partial \mathcal{H}}{\partial \lambda}\right)_{p,q}$. Now putting the two terms together, we have that

$$\dot{I} = \frac{\dot{\lambda}}{2\pi} \left[T(\lambda) \left(\frac{\partial \mathcal{H}}{\partial \lambda}\right)_{p,q} - \int_0^{T(\lambda)} \left(\frac{\partial \mathcal{H}}{\partial \lambda}\right)_{p,q} dt \right]$$

Now we note that if we could pull the integrand out of the integral, these two terms would exactly cancel. Up to this point, we have not leveraged the fact that λ is slowly varying (other than assuming that the orbit closes to good approximation). Now we argue that in the regime where λ is varying adiabatically, these two terms approximately cancel.

Let us compute the change in I over one period:

$$\begin{aligned}
\Delta I &= \int_0^{T(\lambda)} \dot{I}\, dt \\
&= \int_0^{T(\lambda)} dt\, \frac{\dot{\lambda}}{2\pi} \left[T(\lambda) \left(\frac{\partial \mathcal{H}}{\partial \lambda}\right)_{p,q} - \int_0^{T(\lambda)} \left(\frac{\partial \mathcal{H}}{\partial \lambda}\right)_{p,q} dt \right] \\
&= \frac{T(\lambda)}{2\pi} \left[\int_0^{T(\lambda)} dt\, \frac{d\lambda}{dt} \left(\frac{\partial \mathcal{H}}{\partial \lambda}\right)_{p,q} - \frac{1}{T(\lambda)} \int_0^{T(\lambda)} dt \frac{d\lambda}{dt} \int_0^{T(\lambda)} dt \left(\frac{\partial \mathcal{H}}{\partial \lambda}\right)_{p,q} \right] \\
&= \frac{T(\lambda)}{2\pi} \left[\int_0^{T(\lambda)} dt\, \frac{d\lambda}{dt} \left(\frac{\partial \mathcal{H}}{\partial \lambda}\right)_{p,q} - \frac{\Delta\lambda}{T(\lambda)} \int_0^{T(\lambda)} \left(\frac{\partial \mathcal{H}}{\partial \lambda}\right)_{p,q} dt \right] \\
&= \frac{T(\lambda)}{2\pi} \int_0^{T(\lambda)} dt \left(\frac{\partial \mathcal{H}}{\partial \lambda}\right)_{p,q} \left(\frac{d\lambda}{dt} - \frac{\Delta\lambda}{T(\lambda)}\right)
\end{aligned}$$

We now argue that if we are changing λ slowly and smoothly, $\frac{d\lambda}{dt} - \frac{\Delta\lambda}{T(\lambda)} = 0$ over the timescale of one period. We can work through this by Taylor expanding:

$$\Delta\lambda = \frac{d\lambda}{dt}\Delta t + \mathcal{O}\left(\ddot{\lambda}\right)$$

If we look at the condition for adiabaticity, we have that

$$\frac{\dot{\lambda}}{\lambda} \ll \frac{1}{T}$$

Because of this, the higher order terms in the Taylor expansion do not contribute, and thus $\frac{d\lambda}{dt} = \frac{\Delta\lambda}{T(\lambda)}$ (since ΔT over one period is T), and thus the integrand is zero. From this, we have shown that $I = \frac{1}{2\pi} \oint p\, dq$ is a constant of the motion, and is therefore an adiabatic invariant.

3 Electrostatics

We begin our discussion of electrodynamics with electrostatics. The force between two charges satisfies Coulomb's Law:

$$\boldsymbol{F} = \frac{qq_1}{4\pi\varepsilon_0}\frac{\boldsymbol{x}-\boldsymbol{x}_1}{|\boldsymbol{x}-\boldsymbol{x}_1|^3}$$

Where $\boldsymbol{F}$ is the force on q from q_1, and ε_0 is the permittivity of free space. In order to mitigate having to write this prefactor everywhere, we define $k = \frac{1}{4\pi\varepsilon_0}$, which is 9×10^9 Newton meters squared per Coulomb squared.

3.1 Electric Field and Scalar Potential

We now introduce the electric field. Consider a system of n charges, $q_1, \ldots, q_n$, located at $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n$. The force of these n charges on a charge q at location $\boldsymbol{x}$ is given by $\boldsymbol{F} = q\boldsymbol{E}(\boldsymbol{x})$, where

$$\boldsymbol{E}(\boldsymbol{x}) = \sum_i^n \frac{1}{4\pi\varepsilon_0} q_i \frac{\boldsymbol{x}-\boldsymbol{x}_i}{|\boldsymbol{x}-\boldsymbol{x}_i|^3}$$

Note that there is no force exerted on q by itself (there are no self forces). Suppose we have a continuous charge distribution:

$$\boldsymbol{E}(\boldsymbol{x}) = \frac{1}{4\pi\varepsilon_0}\int d^3\boldsymbol{x}'\,\rho(\boldsymbol{x}')\,\frac{\boldsymbol{x}-\boldsymbol{x}'}{|\boldsymbol{x}-\boldsymbol{x}'|^3}$$

Where $\rho(\boldsymbol{x})$ is the density of charge at point $\boldsymbol{x}$. For discrete charges, $\rho(\boldsymbol{x})$ is a sum of delta functions:

$$\rho\boldsymbol{x} = \sum_i q_i \delta^3(\boldsymbol{x}-\boldsymbol{x}_i)$$

Now let us show that the electric field can be written as the gradient of a scalar potential:

$$\boldsymbol{E} = -\boldsymbol{\nabla}\phi$$

Where we define the scalar potential as:

$$\phi(\boldsymbol{x}) = \frac{1}{4\pi\varepsilon_0}\int d^3\boldsymbol{x}'\frac{\rho(\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|}$$

Note that everything in the scalar potential depends on $\boldsymbol{x}'$, except for the denominator, so what we really care about is

$$\boldsymbol{\nabla}\frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|} = \boldsymbol{\nabla}\frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}$$

Let us consider only the x component:

$$\frac{\partial}{\partial x}\frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}} = -\frac{1}{2}\frac{2(x-x')}{\left[(x-x')^2+(y-y')^2+(z-z')^2\right]^{3/2}}$$

$$= -\frac{x - x'}{|\boldsymbol{x} - \boldsymbol{x}'|^3}$$

Repeating this for the other two coordinates, we see that:

$$\boldsymbol{\nabla} \frac{1}{|\boldsymbol{x} - \boldsymbol{x}'|} = -\frac{\boldsymbol{x} - \boldsymbol{x}'}{|\boldsymbol{x} - \boldsymbol{x}'|^3}$$

Thus, we have that the gradient of the scalar potential is:

$$\begin{aligned} -\boldsymbol{\nabla}\phi(\boldsymbol{x}) &= -\frac{1}{4\pi\varepsilon_0} \int d^3\boldsymbol{x}'\, \rho(\boldsymbol{x}')\, \boldsymbol{\nabla} \left[\frac{1}{|\boldsymbol{x} - \boldsymbol{x}'|}\right] \\ &= \frac{1}{4\pi\varepsilon_0} \int d^3\boldsymbol{x}'\, \rho(\boldsymbol{x}')\, \frac{\boldsymbol{x} - \boldsymbol{x}'}{|\boldsymbol{x} - \boldsymbol{x}'|^3} \\ &= \boldsymbol{E}(\boldsymbol{x}) \end{aligned}$$

We see that we can indeed find $\boldsymbol{E}$ from a scalar-valued potential function. This means that we can contain all of the information about the electric field from just this one function, which simplifies things.

Since the curl of a gradient is always zero:

$$\boldsymbol{\nabla} \times \boldsymbol{E} = 0$$

By Stoke's theorem, integrating the electric field around a closed path will give zero:

$$\oint \boldsymbol{E} \cdot d\boldsymbol{l} = 0$$

From this, we find that the integral between two points is independent of the path (since we can choose two paths between a and b, and the total integral must be equal to zero, so any two paths must cancel out exactly):

$$\int_{\boldsymbol{x}_1}^{\boldsymbol{x}_2} \boldsymbol{E} \cdot d\boldsymbol{l} \text{ is path independent}$$

3.2 Charge in an Electric Field

Now let us consider the energy of a charge in an electric field. Consider moving a charge q from $\boldsymbol{x}_1$ to $\boldsymbol{x}_2$ quasistatically, very slowly, so there is no change in kinetic energy. The work done to move the charge is given by:

$$W = \int_{\boldsymbol{x}_1}^{\boldsymbol{x}_2} d\boldsymbol{x} \cdot \boldsymbol{F}_{\text{ext}}$$

The external force is just strong enough to overcome the electric field, so we say that it is equal and opposite to the force due to the electric field, $\boldsymbol{F}_{\text{ext}} = q\boldsymbol{\nabla}\phi$:

$$\begin{aligned} W &= \int_{\boldsymbol{x}_1}^{\boldsymbol{x}_2} d\boldsymbol{x} \cdot F_{\text{ext}} \\ &= q \int_{\boldsymbol{x}_1}^{\boldsymbol{x}_2} d\phi(\boldsymbol{x}) \end{aligned}$$

$$= q\left[\phi\left(\boldsymbol{x}_2\right) - \phi\left(\boldsymbol{x}_1\right)\right]$$

Where we have used the fact that $\boldsymbol{\nabla}\phi \cdot d\boldsymbol{x} = d\phi$, which can be shown by expanding the gradient:

$$\boldsymbol{\nabla}\phi = \frac{\partial\phi}{\partial x}\hat{x} + \frac{\partial\phi}{\partial y}\hat{y} + \frac{\partial\phi}{\partial z}\hat{z}$$

From this, we see that multiplying by the vector $d\boldsymbol{x}$, we get $d\phi\left(\boldsymbol{x}\right)$.

If the electric field vanishes at infinity, we can set the potential at infinity to 0. If we do this, then the potential energy of a charge at $\boldsymbol{x}$ is given by $q\phi\left(\boldsymbol{x}\right)$.

3.3 Poisson's Equation

Consider the divergence of $\boldsymbol{E}$:

$$\boldsymbol{\nabla}\cdot\boldsymbol{E} = \frac{1}{4\pi\varepsilon_0}\int d^3\boldsymbol{x}'\rho\left(\boldsymbol{x}'\right)\boldsymbol{\nabla}\cdot\left(\frac{\boldsymbol{x}-\boldsymbol{x}'}{|\boldsymbol{x}-\boldsymbol{x}'|^3}\right)$$

There is a useful identity that we can apply here:

$$\boldsymbol{\nabla}\cdot\left(\psi\boldsymbol{A}\right) = \boldsymbol{\nabla}\psi\cdot\boldsymbol{A} + \psi\boldsymbol{\nabla}\cdot\boldsymbol{A}$$

We can apply this identity, with $\psi = \frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|^3}$, and $\boldsymbol{A} = \boldsymbol{x}-\boldsymbol{x}'$:

$$\begin{aligned}
\boldsymbol{\nabla}\cdot\boldsymbol{E} &= \frac{1}{4\pi\varepsilon_0}\int d^3\boldsymbol{x}'\,\rho\left(\boldsymbol{x}'\right)\left[\frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|^3}\boldsymbol{\nabla}\cdot\left(\boldsymbol{x}-\boldsymbol{x}'\right) + \left(\boldsymbol{x}-\boldsymbol{x}'\right)\cdot\boldsymbol{\nabla}\left(\frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|^3}\right)\right] \\
&= \frac{1}{4\pi\varepsilon_0}\int d^3\boldsymbol{x}'\rho\left(\boldsymbol{x}'\right)\left[\frac{3}{|\boldsymbol{x}-\boldsymbol{x}'|^3} + \left(\boldsymbol{x}-\boldsymbol{x}'\right)\cdot\left(-\frac{3}{|\boldsymbol{x}-\boldsymbol{x}'|^2}\boldsymbol{\nabla}\left(\frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|}\right)\right)\right] \\
&= \frac{1}{4\pi\varepsilon_0}\int d^3\boldsymbol{x}'\rho\left(\boldsymbol{x}'\right)\left[\frac{3}{|\boldsymbol{x}-\boldsymbol{x}'|^3} + \left(\boldsymbol{x}-\boldsymbol{x}'\right)\cdot\left(\frac{3}{|\boldsymbol{x}-\boldsymbol{x}'|^2}\frac{\boldsymbol{x}-\boldsymbol{x}'}{|\boldsymbol{x}-\boldsymbol{x}'|^3}\right)\right] \\
&= \frac{1}{4\pi\varepsilon_0}\int d^3\boldsymbol{x}'\rho\left(\boldsymbol{x}'\right)\left(\frac{3}{|\boldsymbol{x}-\boldsymbol{x}'|^3} - \frac{3}{|\boldsymbol{x}-\boldsymbol{x}'|^3}\right) \\
&= 0
\end{aligned}$$

Where we have used the fact that $\boldsymbol{\nabla}\cdot\left(\boldsymbol{x}-\boldsymbol{x}'\right) = \boldsymbol{\nabla}\cdot\boldsymbol{x} = 3$, and the fact that $\boldsymbol{\nabla}\left(\psi^3\right) = 3\psi^2\boldsymbol{\nabla}\psi$, and we choose $\psi = 1/(\boldsymbol{x}-\boldsymbol{x}')$.

However, we know that the divergence of $\boldsymbol{E}$ is not zero, so there is a loophole that we have to fix. What we have neglected is the singular point $\boldsymbol{x} = \boldsymbol{x}'$. Consider the integral (note that we are integrating wrt $\boldsymbol{x}$, and not $\boldsymbol{x}'$)

$$I = \int_V d^3\boldsymbol{x}\boldsymbol{\nabla}\cdot\left(\boldsymbol{\nabla}\frac{1}{|\boldsymbol{x}|-\boldsymbol{x}'}\right)$$

Where V is a spherical volume centered around $\boldsymbol{x}'$. If $\boldsymbol{x}$ is a point on the surface of the sphere, then $\boldsymbol{x}-\boldsymbol{x}'$ is the unit vector $\hat{n}$ pointing from the center of the sphere to the point on the sphere, times the radius R of the sphere, $\boldsymbol{x}-\boldsymbol{x}' = R\hat{n}$.

Let us apply the divergence theorem to this volume integral, converting it to a surface integral:

$$\begin{aligned} I &= \int_V d^3\boldsymbol{x} \boldsymbol{\nabla} \cdot \left(\boldsymbol{\nabla} \frac{1}{|\boldsymbol{x}| - \boldsymbol{x}'} \right) \\ &= \int_S ds\, \hat{n} \cdot \boldsymbol{\nabla} \left(\frac{1}{|\boldsymbol{x} - \boldsymbol{x}'|} \right) \end{aligned}$$

Where S is the spherical shell centered at $\boldsymbol{x}'$, and $\hat{n}$ is a unit vector pointing in the direction of the element ds. We now note that $\hat{n} = \frac{\boldsymbol{x} - \boldsymbol{x}'}{R}$, and the radius must be $|\boldsymbol{x} - \boldsymbol{x}'|$. If we now take the gradient:

$$\begin{aligned} I &= \int_S ds\, \hat{n} \cdot \boldsymbol{\nabla} \left(\frac{1}{|\boldsymbol{x} - \boldsymbol{x}'|} \right) \\ &= -\int_S ds\, \hat{n} \cdot \frac{\boldsymbol{x} - \boldsymbol{x}'}{|\boldsymbol{x} - \boldsymbol{x}'|^3} \\ &= -\int_S ds\, \frac{1}{|\boldsymbol{x} - \boldsymbol{x}'|^2} \\ &= -\int_S ds\, \frac{1}{R^2} \\ &= -4\pi \end{aligned}$$

Since the result is independent of R, we must have

$$\boldsymbol{\nabla}^2 \left(\frac{1}{|\boldsymbol{x} - \boldsymbol{x}'|} \right) = -4\pi \delta^3 \left(\boldsymbol{x} - \boldsymbol{x}' \right)$$

This explains why we got 0 in our first derivation, since this is zero at all points other than $\boldsymbol{x} = \boldsymbol{x}'$, and we were not careful about that singular point.

If we now look at $\boldsymbol{\nabla} \cdot \boldsymbol{E}$:

$$\begin{aligned} \boldsymbol{\nabla} \cdot \boldsymbol{E} &= \frac{1}{4\pi\varepsilon_0} \int d^3\boldsymbol{x}'\, \rho\left(\boldsymbol{x}'\right) 4\pi \delta^3 \left(\boldsymbol{x} - \boldsymbol{x}' \right) \\ &= \frac{\rho\left(\boldsymbol{x}\right)}{\varepsilon_0} \end{aligned}$$

Which is the differential form of Gauss's law, as we expect:

$$\boldsymbol{\nabla} \cdot \boldsymbol{E} = \frac{\rho}{\varepsilon_0}$$

We also know that $\boldsymbol{E} = -\boldsymbol{\nabla}\phi$, so we have that

$$\boldsymbol{\nabla}^2 \phi = -\frac{\rho}{\varepsilon_0}$$

For the special case of $\rho = 0$, we recover Laplace's equation:

$$\boldsymbol{\nabla}^2 \phi = 0$$

3.4 Gauss's Law

Let us now derive the integral form of Gauss's Law. We begin with the differential form, which we derived above:

$$\boldsymbol{\nabla} \cdot \boldsymbol{E} = \frac{\rho}{\varepsilon_0}$$

Integrating this over some arbitrary closed volume:

$$\int_V d^3\boldsymbol{x}\, \boldsymbol{\nabla} \cdot \boldsymbol{E} = \frac{1}{\varepsilon_0} \underbrace{\int_V d^3\boldsymbol{x}\, \rho\left(\boldsymbol{x}\right)}_{Q_{\text{enc}}}$$

The right hand integral is the total charge enclosed in V. The right side can be written as a surface integral using the divergence theorem:

$$\int d^3\boldsymbol{x}\, \boldsymbol{\nabla} \cdot \boldsymbol{E} = \int_S ds\ \left(\hat{n} \cdot \boldsymbol{E}\right)$$

Where $\hat{n}$ is the outward normal vector to the surface element ds. This integral is the total flux of $\boldsymbol{E}$ through the closed surface S, and we have the relation:

$$\int_S ds\, \boldsymbol{E} \cdot \hat{n} = \frac{Q_{\text{enc}}}{\varepsilon_0} \tag{8}$$

This is the integral form of Gauss's Law, it relates the flux of the electric field through a surface to the total charge enclosed by the surface.

Let us do a couple of examples. Suppose we want to determine the electric field of a point charge q using Gauss's Law.

We consider a sphere around the point charge, of radius R. The charge enclosed is q.

$$\oint \boldsymbol{E} \cdot d\hat{n} = \frac{q}{\varepsilon_0}$$

The magnitude of the electric field will be the same for all points on the surface of our sphere, so $\boldsymbol{E} = E\hat{r}$. The unit vector to any point on the sphere is also in the radial direction, $\hat{n} = \hat{r}$. Applying this to the left hand side:

$$\begin{aligned}
\oint \boldsymbol{E} \cdot d\hat{n} &= \frac{q}{\varepsilon_0} \\
\oint E\hat{r} \cdot d\hat{r} &= \frac{q}{\varepsilon_0} \\
4\pi R^2 E &= \frac{q}{\varepsilon_0} \\
E &= \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}
\end{aligned}$$

From this, we obtain the expected:

$$\boldsymbol{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r}$$

Now let us consider a slightly more complicated example. Consider the electric field due to the surface charge of a conductor. The first thing to note is that the electric field inside of a conductor is zero, since the charges inside of a conductor are free to move, and thus move to make the electric field zero. Essentially, we have no electric field or free charges inside a conductor. This implies that the charge density inside a conductor must be 0, since $\boldsymbol{\nabla} \cdot \boldsymbol{E} = \rho$. This restricts any free charges to the surface.

We define the surface charge density $\sigma(\boldsymbol{x})$ as the charge on an area element, over the area of the element, where the area is infinitesimal.

We have now set up the problem, let us now find the electric field at the surface due to this surface charge density. To do this, we consider a "pillbox" of area ΔS and infinitesimal height at the surface.

We assume that the area of the surface that we contain is small enough that the surface charge density cannot vary that much:

$$Q_{\text{enc}} = \sigma \Delta S$$

Writing down Gauss's Law:

$$\begin{aligned} \oint \boldsymbol{E} \cdot \hat{n}\, ds &= \frac{Q}{\varepsilon_0} \\ &= \frac{1}{\varepsilon_0} (\sigma \Delta S) \end{aligned}$$

Note that the height of the pillbox is infinitesimal even compared to the area of the pillbox, which is also infinitesimal. The bottom surface of the pillbox does not contribute to the flux, because it is inside the conductor, and $\boldsymbol{E} = 0$. The sides don't contribute because the height of the pillboxe is negligible, and thus only the top contributes to the surface integral. Thus we are left with

$$\boldsymbol{E} \cdot \hat{n} \Delta S = \frac{1}{\varepsilon_0} \sigma \Delta S$$

The normal component of $\boldsymbol{E}$ is given by

$$\boldsymbol{E} \cdot \hat{n} = \frac{\sigma}{\varepsilon_0}$$

What about the tangential component of $\boldsymbol{E}$?

Consider an infinitesimal rectangular loop. Recall that the integral of the electric field along the entire loop must be zero, $\oint \boldsymbol{E} \cdot d\boldsymbol{l} = 0$. We can neglect the sides of the loop, since we make them infinitesimally small compared to the width of the loop. The contribution from the bottom vanishes because it is inside the conductor, where $\boldsymbol{E} = 0$. Thus we have that the overall integral, which is equal to just the contribution from the part outside of the conductor, must be zero, and thus the tangential electric field outside of the conductor must also be zero. The tangential field is the field *parallel* to the surface of the conductor.

Thus we have shown that near the surface of a conductor, the electric field is solely perpendicular to the surface, and is given by

$$\boldsymbol{E} \cdot \hat{n} = \frac{\sigma}{\varepsilon_0}$$

3.5 Uniqueness of Solutions of Poisson's Equations

We often have to solve Poisson's equation in a region with boundaries. We now want to show that if Dirichlet or Nuemann boundary conditions are imposed at the boundary, the solution that we obtain is unique.

We go about proving this by contradiction. Poisson's equation is of the form:

$$\boldsymbol{\nabla}^2\phi = -\frac{\rho}{\varepsilon_0}$$

Suppose that this has two solutions, ϕ_1 and ϕ_2 such that $\phi_1 \neq \phi_2$. We also suppose that these two solutions both satisfy the boundary conditions of the system:

$$\boldsymbol{\nabla}^2\phi_1 = -\frac{\rho}{\varepsilon_0}$$
$$\boldsymbol{\nabla}^2\phi_2 = -\frac{\rho}{\varepsilon_0}$$

Now consider the function $U = \phi_1 - \phi_2$. This necessarily satisfies Laplace's equation:

$$\boldsymbol{\nabla}^2 U = 0$$

We can now consider the quantity:

$$\int_V d^3\boldsymbol{x}\, U\boldsymbol{\nabla}^2 U$$

Since $\boldsymbol{\nabla}^2 U = 0$, this quantity must also be zero. We can rewrite the integrand:

$$\int_V d^3\boldsymbol{x}\, U\boldsymbol{\nabla}^2 U = \int_V d^3\boldsymbol{x}\, \left[\boldsymbol{\nabla}\cdot(U\boldsymbol{\nabla}U) - |\boldsymbol{\nabla}U|^2\right]$$

Now setting this equal to zero and splitting the integral into two parts:

$$\int_V d^3\boldsymbol{x}\, \boldsymbol{\nabla}\cdot(U\boldsymbol{\nabla}U) = \int_V d^3\boldsymbol{x}\, |\boldsymbol{\nabla}U|^2$$

Now applying the divergence theorem to the left side:

$$\int_S ds\, \hat{n}\cdot(U\boldsymbol{\nabla}U) = \int ds\, U\,(\hat{n}\cdot\boldsymbol{\nabla}U)$$

For Dirichlet boundary conditions, the value of ϕ is specified at the boundary S. If we impose these conditions, then U must vanish at the boundary, so

$$\int_V d^3\boldsymbol{x}\, |\boldsymbol{\nabla}U|^2 = 0$$

This implies that U is a constant throughout the volume V, but since U vanishes at the boundary, this means that U is zero everywhere, which means that $\phi_1 = \phi_2$, giving us a contradiction.

In the case of Neumann boundary conditions, the value of $\boldsymbol{\nabla}\phi\cdot\hat{n}$ is specified on the boundary. In this case, $\hat{n}\cdot\boldsymbol{\nabla}U = 0$ on the boundary, and by the same argument as in the previous case, U must be zero everywhere, once again providing us a contradiction. Thus we have that the scalar potential ϕ is unique up to an additive constant.

3.6 Electrostatic Energy

Consider a charge q, located at $\boldsymbol{x}_1$. The potential at a point $\boldsymbol{x}$ due to $\boldsymbol{x}_1$ is given by

$$\phi = \frac{1}{4\pi\varepsilon_0}\frac{q_1}{|\boldsymbol{x}-\boldsymbol{x}_1|}$$

Now suppose we move another charge from ∞ to $\boldsymbol{x}_2$. The potential energy is

$$\begin{aligned} W &= q_2\phi\left(\boldsymbol{x}_1\right) \\ &= \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{|\boldsymbol{x}_2-\boldsymbol{x}_1|} \end{aligned}$$

If we now bring in an additional charge from infinity to $\boldsymbol{x}_3$:

$$W = \frac{1}{4\pi\varepsilon_0}\left[\frac{q_1 q_2}{|\boldsymbol{x}_1-\boldsymbol{x}_2|}+\frac{q_2 q_3}{|\boldsymbol{x}_2-\boldsymbol{x}_3|}+\frac{q_3 q_1}{|\boldsymbol{x}_3-\boldsymbol{x}_1|}\right]$$

This immediately generalizes to an arbitrary number of charges:

$$W = \sum_{i=1}^{n}\sum_{j<i}\frac{1}{4\pi\varepsilon_0}\frac{q_i q_j}{|\boldsymbol{x}_i-\boldsymbol{x}_j|}$$

Which we can rewrite as

$$W = \frac{1}{2}\sum_{i\neq j}\frac{q_i q_j}{|\boldsymbol{x}_j-\boldsymbol{x}_i|}\frac{1}{4\pi\varepsilon_0}$$

The condition $i \neq j$ removes the self-energy terms.

For a continuous charge distribution $\rho\left(\boldsymbol{x}\right)$:

$$\begin{aligned} W &= \frac{1}{2}\int d^3\boldsymbol{x}\int d^3\boldsymbol{x}'\,\frac{\rho\left(\boldsymbol{x}\right)\rho\left(\boldsymbol{x}'\right)}{|\boldsymbol{x}-\boldsymbol{x}'|}\frac{1}{4\pi\varepsilon_0} \\ &= \frac{1}{2}\int d^3\boldsymbol{x}\,\rho\left(\boldsymbol{x}\right)\phi\left(\boldsymbol{x}\right) \end{aligned}$$

Where we have substituted in the definition of the potential. If we take this expression and insert a rewritten form of Poisson's equation:

$$W = -\frac{1}{2}\varepsilon_0\int d^3\boldsymbol{x}\,\phi\left(\boldsymbol{x}\right)\boldsymbol{\nabla}^2\phi\left(\boldsymbol{x}\right)$$

If we integrate this by parts, while assuming that the field vanishes at infinity:

$$\begin{aligned} W &= \frac{1}{2}\varepsilon_0\int d^3\boldsymbol{x}\,|\boldsymbol{\nabla}\phi|^2 \\ &= \frac{1}{2}\varepsilon_0\int d^3\boldsymbol{x}\,|\boldsymbol{E}|^2 \end{aligned}$$

Which gives us the energy density of an electrostatic field:

$$w = \frac{1}{2}\varepsilon_0|\boldsymbol{E}|^2$$

Note that the energy density is positive definite, while the expression for discrete charges is not, because the energy density includes the self-energy.

3.7 Force on a Conducting Sphere

We can obtain the force on a conducting surface in two differeny ways. The first is to look at the energy change from a virtual displacement, and the second is directly from the electric field.

Let us first consider the energy change from a virtual displacement. The energy density is $w = \frac{1}{2}\varepsilon_0|\boldsymbol{E}|^2$. At the surface of the conductor, $\boldsymbol{E} = \frac{\sigma}{\varepsilon_0}$, and is directed normal to the surface. Consider a smalll displacement Δx of an element ΔA of the conducting surface, in a direction normal to the surface at that point. The change in energy will be given by:

$$\begin{aligned}\Delta w &= -\frac{\varepsilon_0}{2}|\boldsymbol{E}|^2\Delta A\Delta x \\ &= -\frac{1}{2\varepsilon_0}\sigma^2\Delta A\Delta x\end{aligned}$$

The force is given by $F = -\frac{\partial w}{\partial x}$, and therefore we have that

$$F = \sigma^2\Delta A\frac{1}{2\varepsilon_0}$$

and the force is directed outwards from the surface of the conductor. We can also compute the force per unit area:

$$f = \frac{\sigma^2}{2\varepsilon_0}$$

The net force on the conductor can be obtained by integrating this over the surface.

The second method is to determine the force on the area element ΔA:

$$\boldsymbol{F} = (\sigma\Delta a)\,\boldsymbol{E}_{\text{ext}}$$

Where the external electric field $\boldsymbol{E}_{\text{ext}}$ is the electric field of the conductor after the contribution from the element ΔA has been removed. This contribution is given by:

$$E_{\text{self}} = \frac{\sigma}{2\varepsilon_0}$$

The total electric field is the sum of these two fields:

$$E_{\text{total}} = E_{\text{self}} + E_{\text{ext}}$$

From this, we find that $E_{\text{ext}} = \frac{\sigma}{2\varepsilon_0}$. Thus we have that

$$\begin{aligned}f &= \frac{F}{\Delta A} \\ &= \frac{1}{2\varepsilon_0}\sigma^2\end{aligned}$$

We see that we arrive at the same result as the energy argument. Always remember that when you are calculating forces directly, we must remove the contribution from the self force, since nothing can accelerate itself.

3.8 Capacitance

Consider an isolate conductor carrying a charge Q. The capacitance C is defined as:

$$C = \frac{Q}{\phi}$$

Where ϕ is the potential. If the capacitance is large, the conductor can store a lot of charge, even though the potential only changes by a small amount. The value of C depends only on the geometry of the conductor.

Capacitance for the case of two conductors carrying equal and opposite charges is defined as the ratio of the charge on one conductor to the potential difference between them:

$$C = \frac{Q}{\Delta\phi}$$

3.9 Boundary Value Problems

Consider a grounded conducting sphere of radius a (so the surface potential is zero). We have a charge q outside the sphere at some point $\boldsymbol{y}$ relative to the center of the sphere. We want to find the potential outside the sphere, at some arbitrary point $\boldsymbol{x}$.

To do this, we use the method of images, we want to place a charge q' somewhere and replace the sphere, which will simplify the calculation. We have two constraints on where we can place the charge. The first is that it lies along $\boldsymbol{y}$, since that is the only axis that is important to the problem, everything else is symmetric. The second constraint is that the image charge must be placed inside the sphere.

In general, the potential $\phi(\boldsymbol{x})$ at $\boldsymbol{x}$ will be:

$$\begin{aligned}\phi(\boldsymbol{x}) &= \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{|\boldsymbol{x}-\boldsymbol{y}|} + \frac{q'}{|\boldsymbol{x}-\boldsymbol{y}|}\right] \\ &= \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{|x\hat{n} - y\hat{n}'|} + \frac{q'}{|x\hat{n} + y\hat{n}'|}\right]\end{aligned}$$

Where we have replaced the magnitude of vectors with their non-boldface forms, and $\hat{n}$ is the unit vector from the center of the sphere to $\boldsymbol{x}$, and $\hat{n}'$ is the unit vector from the center to the external charge.

We now introduce the boundary condition constraint, $\phi(x = a) = 0$, the potential at the surface of the sphere must be zero:

$$\phi(x=a) = \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{a|\hat{n} - \frac{y}{a}\hat{n}'|} + \frac{q'}{y'|\hat{n}' - \frac{a}{y}\hat{n}|}\right]$$

In order for this to be zero, we make the choice that $\frac{q}{a} = -\frac{q'}{y'}$, and $\frac{y}{a} = \frac{a}{y'}$. This is how we set our choice for where the image charge is placed. This gives us the general solution:

$$\phi(\boldsymbol{x}) = \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{|\boldsymbol{x}-\boldsymbol{y}|} - \frac{qa}{y}\frac{1}{|\boldsymbol{x} - \frac{a^2}{y^2}\boldsymbol{y}|}\right]$$

From this, we could find the electric field, using $\boldsymbol{E} = -\boldsymbol{\nabla}\phi$.

This is the simplest case of the method of images.

3.10 Green's Function Method

The Green's function method allows us to more easily solve boundary value problems, as well as expand the set of boundary values that we can deal with.

We begin with an ordinary differential equation:

$$y'' + y = f(x)$$

We set our boundary conditions such that $y = 0$ at $x = 0$, and $y = 0$ at $x = \frac{\pi}{2}$.

The strategy to solve this is to introduce a function $g(x, x')$, that satisfies a slightly different version of this equation, where we introduce a new variable x':

$$\frac{d^2G}{dx^2} + G = \delta(x - x') \qquad G(x, x')\,|_{x=0,\pi/2} = 0$$

We then use this function to solve for $y(x)$:

$$y(x) = \int_0^{\pi/2} G(x, x')\, f(x')\, dx'$$

We can check that this is valid:

$$\begin{aligned} y'' + y &= \int_0^{\pi/2} \left[\frac{d^2G}{dx^2} + G\right] f(x')\, dx' \\ &= \int_0^{\pi/2} \delta(x - x')\, f(x')\, dx' \\ &= f(x) \end{aligned}$$

We can also see that inserting $x = 0$ or $x = \frac{\pi}{2}$ matches the boundary conditions.

Essentially, we determine the function $G(x, x')$ for all points x', and then aggregating all of these by integrating over x' to obtain $y(x)$, and scaling by the value of the Green's function at each point. This is essentially solving the ODE at every point, and then superimposing the different solutions together.

Thus, in order to obtain $y(x)$, we need to find $G(x, x')$. The claim is that finding G is easier than solving the original ODE.

We begin with our differential equation:

$$\frac{d^2G}{dx^2} + G = \delta(x - x')$$

We know that away from x', this is a homogenous equation:

$$\frac{d^2G}{dx^2} + G = 0$$

Which has the harmonic oscillator solution, $G = A\sin(x) + B\cos(x)$. Thus the only case that we need to worry about is the case where $x = x'$. Since we have a discontinuity at $x = x'$, we treat the Green's function as a piecewise function:

$$G = \begin{cases} G_< = A_< \sin x + B_< \cos x & x < x' \\ G_> = A_> \sin x + B_> \cos x & x > x' \end{cases}$$

We now apply the boundary condition that for $x = 0$, $G_<(0, x') = 0$, which gives us that $B_< = 0$. We apply the other boundary condition, at $x = \frac{\pi}{2}$, we have $G_> (\pi/2, x') = 0$, so $A_> = 0$.

Now we enforce continuity, so $G_< (x', x') = G_> (x', x')$:

$$A_< \sin\left(x'\right) = B_> \cos\left(x'\right)$$

To get a second equation, we integrate our original equation across a band around x':

$$\begin{aligned}
\int_{x'-\varepsilon}^{x'+\varepsilon} \left[\frac{d^2G}{dx^2} + G\right] \, dx &= \int_{x'-\varepsilon}^{x'+\varepsilon} \delta\left(x - x'\right) \, dx \\
\frac{dG}{dx}|_{x=x'+\varepsilon} - \frac{dG}{dx}|_{x=x'-\varepsilon} &= 1 \\
-B_> \sin\left(x'\right) - A_< \cos\left(x'\right) &= 1
\end{aligned}$$

Where the integral of G vanishes over this small region because the contribution will be of order ε.

We now have two equations of the two constants, and thus we can determine them:

$$A_< = -\cos\left(x'\right) \qquad B_> = -\sin\left(x'\right)$$

This gives us the Green's function:

$$G\left(x, x'\right) = \begin{cases} -\sin x \cos x' & x < x' \\ -\cos x \sin x' & x > x' \end{cases}$$

We can now compute $y\left(x\right)$:

$$\begin{aligned}
y\left(x\right) &= \int_0^{\pi/2} dx' \, G\left(x, x'\right) f\left(x'\right) \\
&= \int_0^x dx' \, G_>\left(x, x'\right) f\left(x'\right) + \int_x^{\pi/2} dx' \, G_<\left(x, x'\right) f\left(x'\right)
\end{aligned}$$

Which is the solution to our differential equation.

Now let us apply this method to electrostatics problems.

Consider a point charge q. We claim that the potential from this point charge can be interpreted as the Green's function of $\boldsymbol{\nabla}^2$. We can see this a little more clearly by looking at Poisson's equation:

$$\begin{aligned}
\boldsymbol{\nabla}^2 \phi &= -\frac{\rho}{\varepsilon_0} \\
&= -\frac{q}{\varepsilon_0} \delta^3\left(\boldsymbol{x} - \boldsymbol{x}'\right)
\end{aligned}$$

The potential from a point charge is given by:

$$\phi\left(\boldsymbol{x}\right) = \frac{1}{4\pi\varepsilon_0} \frac{1}{|\boldsymbol{x} - \boldsymbol{x}'|}$$

We want our Green's function to satisfy:

$$\boldsymbol{\nabla}^2 G = -4\pi\delta^3\left(\boldsymbol{x}-\boldsymbol{x}'\right)$$

Where the -4π is convention. If we take this convention, we have that

$$\begin{aligned} G\left(\boldsymbol{x},\boldsymbol{x}'\right) &= \frac{4\pi\varepsilon_0}{q}\phi\left(\boldsymbol{x}\right) \\ &= \frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|} \end{aligned}$$

Now let us consider a general potential, generated by some general charge distribution. We can see that we can use the Green's function to generate this out of a superposition of point charge potentials:

$$\begin{aligned} \phi\left(\boldsymbol{x}\right) &= \frac{1}{4\pi}\int d^3\boldsymbol{x}'\, G\left(\boldsymbol{x},\boldsymbol{x}'\right)\frac{\rho\left(\boldsymbol{x}'\right)}{\varepsilon_0} \\ &= \frac{1}{4\pi\varepsilon_0}\int d^3\boldsymbol{x}'\frac{\rho\left(\boldsymbol{x}\right)}{|\boldsymbol{x}-\boldsymbol{x}'|} \end{aligned}$$

Which produces the standard result.

To recap, in the case of the ODE, we solved $y''+y=f\left(x\right)$, and in the case of the Poisson equation, we solved $\boldsymbol{\nabla}^2\phi=-\frac{\rho(x)}{ve_0}$. We then wrote these in terms of a Green's function:

$$G''+G=\delta\left(x-x'\right) \qquad \boldsymbol{\nabla}^2 G=-4\pi\delta^3\left(\boldsymbol{x}-\boldsymbol{x}'\right)$$

We then solved for the Green's functions, which was piecewise in the case of the ODE, and was $1/|\boldsymbol{x}-\boldsymbol{x}'|$ in the electrostatics case. We then used the Green's function to solve for the solution to the differential equation:

$$y=\int dx'\, f\left(x'\right)G\left(x,x'\right) \qquad \frac{1}{4\pi}\int d^3\boldsymbol{x}'\, G\left(\boldsymbol{x},\boldsymbol{x}'\right)\frac{\rho\left(\boldsymbol{x}'\right)}{\varepsilon_0}$$

Recall the case we solved using the method of images, where we have a point charge outside of a grounded conducting sphere:

$$\phi\left(\boldsymbol{x}\right)=\frac{1}{4\pi\varepsilon_0}\left[\frac{q}{|\boldsymbol{x}-\boldsymbol{x}'|}-\frac{aq}{|\boldsymbol{x}'|}\frac{1}{\left|\boldsymbol{x}-\frac{a^2}{|\boldsymbol{x}'|^2}\boldsymbol{x}'\right|}\right]$$

We know that this satisfies the Poisson equation:

$$\boldsymbol{\nabla}^2\phi=-\frac{q}{\varepsilon_0}\delta^3\left(\boldsymbol{x}-\boldsymbol{x}'\right)$$

With $\phi\left(|\boldsymbol{x}|=a\right)=0$.

We choose $q=4\pi\varepsilon_0$ to get the Green's function:

$$G\left(\boldsymbol{x},\boldsymbol{x}'\right)=\left[\frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|}-\frac{a}{|\boldsymbol{x}'|}\frac{1}{\left|\boldsymbol{x}-\frac{a^2}{|\boldsymbol{x}'|^2}\boldsymbol{x}'\right|}\right]$$

Which we can now apply to a generic charge distribution outside the sphere:

$$\phi(\boldsymbol{x}) = \frac{1}{4\pi\varepsilon_0}\int d^3\boldsymbol{x}\, G(\boldsymbol{x},\boldsymbol{x}')\,\rho(\boldsymbol{x}')$$

Essentially, we are able to solve a particular case using a point charge, and then apply it to a general charge distribution in the same situation.

Theorem 3.1. *Green's Theorem For two scalar functions ψ and ϕ:*

$$\int_V \left(\phi\boldsymbol{\nabla}^2\psi - \psi\boldsymbol{\nabla}^2\phi\right)\, d^3\boldsymbol{x} = \oint_S \left[\phi\left(\boldsymbol{\nabla}\psi\cdot\hat{n}\right) - \psi\left(\boldsymbol{\nabla}\phi\cdot\hat{n}\right)\right] ds$$

Proof. We can compute the two terms in the left integral:

$$\phi\boldsymbol{\nabla}^2\psi = \boldsymbol{\nabla}\cdot(\phi\boldsymbol{\nabla}\psi) - \boldsymbol{\nabla}\phi\cdot\boldsymbol{\nabla}\psi$$
$$\psi\boldsymbol{\nabla}^2\phi = \boldsymbol{\nabla}\cdot(\psi\boldsymbol{\nabla}\phi) - \boldsymbol{\nabla}\psi\cdot\boldsymbol{\nabla}\phi$$

We can now subtract these two terms:

$$\phi\boldsymbol{\nabla}^2\psi - \psi\boldsymbol{\nabla}^2\phi = \boldsymbol{\nabla}\cdot(\phi\boldsymbol{\nabla}\psi - \psi\boldsymbol{\nabla}\phi)$$

We can now integrate this over the volume, and then apply the divergence theorem, and we achieve the desired result. □

How do we use this theorem? We choose ϕ to be the potential that we are looking for, and we choose ψ to be the Green's function:

$$\psi = G(x,x')$$

Such that

$$\boldsymbol{\nabla}^2 G = -4\pi\delta^3(\boldsymbol{x}-\boldsymbol{x}')$$

Where the boundary conditions will be specified later. Using Green's theorem:

$$\int_V \left[\phi(\boldsymbol{x}')\,\boldsymbol{\nabla}'^2\psi(x') - \psi(\boldsymbol{x}')\,\boldsymbol{\nabla}'^2\phi(\boldsymbol{x}')\right]\, d^3\boldsymbol{x}' = \int_S \left[\phi(\boldsymbol{x})\left(\boldsymbol{\nabla}'\psi(\boldsymbol{x}')\cdot\hat{n}'\right) - \psi(\boldsymbol{x}')\left(\boldsymbol{\nabla}'\phi(\boldsymbol{x}')\cdot\hat{n}'\right)\right]\, ds'$$

From this, we have that the potential is given by

$$\phi(\boldsymbol{x}) = \frac{1}{4\pi\varepsilon_0}\int_V \rho(\boldsymbol{x}')\,G(\boldsymbol{x}',\boldsymbol{x})\, d^3\boldsymbol{x}' + \frac{1}{4\pi}\int_S \left[G(\boldsymbol{x}',\boldsymbol{x})\left(\boldsymbol{\nabla}\phi(\boldsymbol{x}')\cdot\hat{n}'\right) - \phi(\boldsymbol{x}')\left(\boldsymbol{\nabla}G(\boldsymbol{x}',\boldsymbol{x})\cdot\hat{n}'\right)\right]\, ds'$$

This is our key result.

We can now consider the case with Dirichlet boundary conditions, $G_D = 0$ for $x' \in S$, so the volume integral vanishes.

In the case of Neumann boundary conditions, we could guess that $\boldsymbol{\nabla}' G_N(\boldsymbol{x}, \boldsymbol{x}') \cdot \hat{n} = 0$ for $\boldsymbol{x}' \in S$, but this gives us an inconsistent solution. Instead, we choose

$$\boldsymbol{\nabla}'^2 G_N(\boldsymbol{x}, \boldsymbol{x}') \cdot \hat{n}' = -\frac{4\pi}{s}$$

Which gives us

$$\phi(\boldsymbol{x}') = \langle\phi\rangle_S + \frac{1}{4\pi\varepsilon_0}\int_V \rho(\boldsymbol{x}')\, G_N(\boldsymbol{x}, \boldsymbol{x}')\, d^3\boldsymbol{x}' + \frac{1}{4\pi}\oint_S G_N(\boldsymbol{x}', \boldsymbol{x})\left(\boldsymbol{\nabla}'\phi(\boldsymbol{x}') \cdot \hat{n}'\right) ds'$$

Where $\langle\phi\rangle_S$ denotes the average of the value of ϕ on the surface S. If one boundary surface is at ∞, then usually $\langle\phi\rangle_S = 0$, and that term vanishes.

It can be shown that in general, $G(\boldsymbol{x}', \boldsymbol{x}) = G(\boldsymbol{x}, \boldsymbol{x}')$, which is a good way of checking whether the Green's function is correct.

Let us do an example. Consider a grounded, conducting sphere, of radius a, with a thin insulating band around its center (at $z = 0$). The upper hemisphere is kept at a potential $+V$, and the lower hemisphere is kept at potential $-V$. We want to find the potential outside of the sphere.

We are working with Dirichlet boundary conditions, so we know what our potential will look like:

$$\phi(\boldsymbol{x}) = \frac{1}{4\pi\varepsilon_0}\int_V \rho(\boldsymbol{x}')\, G_D(\boldsymbol{x}, \boldsymbol{x}')\, d^3\boldsymbol{x}' - \frac{1}{4\pi}\int \phi(\boldsymbol{x}')\left(\boldsymbol{\nabla}' G_D(\boldsymbol{x}, \boldsymbol{x}') \cdot \hat{n}'\right) ds'$$

Since the sphere is conducting, the charge density is zero, so the first term vanishes. We also replace the unit vector $\hat{n}'$ with the radial component, $-\hat{r}'$ (note that this points inwards).

We know the Green's function, which is the same as the usual conducting sphere, except we rewrite it in spherical coordinates:

$$G_D = \frac{1}{|\boldsymbol{x} - \boldsymbol{x}'|} - \frac{a}{r'\left|\boldsymbol{x} - \frac{a^2}{r'^2}\boldsymbol{x}'\right|}$$

We now put these together to solve for the potential. In spherical coordinates:

$$|\boldsymbol{x} - \boldsymbol{x}'| = \sqrt{r^2 + r'^2 - 2rr'\cos\gamma}$$

Where γ is the angle between the two points, and $\cos\gamma = \hat{r} \cdot \hat{r}'$. We can also use the definition of the radial unit vector:

$$\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos\left(\phi - \phi'\right)$$

We can also compute the derivative of the Green's function with respect to r':

$$\frac{dG_D}{dr'} = \frac{-r' + r\cos\gamma}{\left(r^2 + r'^2 - 2rr'\cos\gamma\right)^{3/2}} + \frac{\frac{r^2 r'}{a^2} - r\cos\gamma}{\left(\frac{r^2 r'^2}{a^2} + a^2 - 2rr'\cos\gamma\right)^{3/2}}$$

Plugging these all in:

$$\phi(\boldsymbol{x}) = \frac{1}{4\pi}\oint_S \phi(a, \theta', \phi')\, \frac{a^2\left(r^2 - a^2\right)}{\left(r^2 + a^2 - 2ar\cos\gamma\right)^{3/2}} d\Omega'$$

$$= \frac{Va^2\left(r^2-a^2\right)}{4\pi a}\int_0^{2\pi} d\phi' \int_0^1 d\cos\theta' \frac{1}{\left(a^2+r^2-2ar\cos\gamma\right)^{3/2}}$$
$$- \frac{Va^2\left(r^2-a^2\right)}{4\pi a}\int_0^{2\pi} d\phi' \int_{-1}^0 d\cos\theta' \frac{1}{\left(a^2-r^2-2ar\cos\gamma\right)^{3/2}}$$

In the case where $r \gg a$, we have that

$$\begin{aligned}\phi(\boldsymbol{x}) &= \frac{Var^2}{4\pi}\frac{3a}{r^4}(2\pi)\cos\theta \\ &= \frac{3}{2}\left(\frac{a}{r}\right)^2 V\cos\theta\end{aligned}$$

Which is just the dipole, which we expect from the long distance limit.

3.11 Orthogonal Basis Functions

When the geometry of a problem is simple, we can use the method of images to obtain the Green's function. In complex geometries, this is often challenging. For example, consider a configuration with two conductors, each of which is an infinite plane. This would require an infinite number of image charges, which would be very tricky. Instead, we would like an alternative method for problems like this, which is the method of orthogonal basis functions. We expand the solution to Lapace's or Poisson's equations in terms of basis functions that are dependent on the geometry of the problem. There are 3 commonly used coordinate systems, Cartesian, cylindrical, and spherical. We will see how to set up the orthogonal basis functions in all of these coordinate systems.

3.11.1 Cartesian Coordinates

Consider a box whose sides and bottom face have potential 0. We have that the bottom of the box lies in the xy plane, and the vertical direction is z. The potential at the top of the box is given by $\Phi = V(x,y)$. The corner opposite the origin is at (a,b,c), giving the box dimensions $a \times b \times c$. We want to solve for the potential anywhere inside the box. We have to solve Laplace's equation in Cartesian coordinates, subject to the boundary conditions that we specified. In Cartesian coordinates, Laplace's equation is given by

$$\frac{\partial^2\Phi}{\partial x^2}+\frac{\partial^2\Phi}{\partial y^2}+\frac{\partial^2\Phi}{\partial z^2}=0$$

We will solve this with separation of variables, we will look for a solution of the form:

$$\hat{\phi}(x,y,z) = X(x)\,Y(y)\,Z(z)$$

We substitute this into our equation, and then divide by $\hat{\phi}$, giving us:

$$\frac{1}{X}\frac{d^2X}{dx^2}+\frac{1}{Y}\frac{d^2Y}{dy^2}+\frac{1}{Z}\frac{d^2Z}{dz^2}=0$$

Where the derivatives are now total derivatives. We can rewrite this:

$$\frac{1}{X}\frac{d^2X}{dx^2}+\frac{1}{Y}\frac{d^2Y}{dy^2}=-\frac{1}{Z}\frac{d^2Z}{dz^2}$$

We see that the left side only depends on x and y, and the right side only depends on z.

Thus we can set the two of these sides to the same constant, which we choose to be $-\gamma^2$. The choice of the sign comes from experience (or a guess).

We now have two differential equations:

$$\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} = -\gamma^2$$
$$\frac{1}{Z}\frac{d^2Z}{dz^2} = \gamma^2$$

We can again separate variables for the x and y equation:

$$\frac{1}{X}\frac{d^2X}{dx^2} = -\gamma^2 - \frac{1}{Y}\frac{d^2Y}{dy^2}$$

This separates into two equations:

$$\frac{1}{X}\frac{d^2X}{dx^2} = -\alpha^2$$
$$\gamma^2 + \frac{1}{Y}\frac{d^2Y}{dy^2} = \alpha^2$$

Defining $\beta^2 = \gamma^2 - \alpha^2$, we have a total of 3 equations:

$$\frac{1}{X}\frac{d^2X}{dx^2} = -\alpha^2$$
$$\frac{1}{Y}\frac{d^2Y}{dy^2} = -\beta^2$$
$$\frac{1}{Z}\frac{d^2Z}{dz^2} = \gamma^2$$

These equations are simple to solve, the x and y equations are just harmonic oscillator equations, so we get sines and cosines, and the z equation gives us hyperbolic sines and cosines.

We have that $X\left(x=0, x=a\right) = 0$, $Y\left(y=0, y=b\right) = 0$, and $Z\left(z=0\right) = 0$. Because of these, we have the following solutions:

$$X\left(x\right) = \sin\left(\frac{n\pi x}{a}\right) \qquad n \in \mathbb{Z}$$
$$Y\left(y\right) = \sin\left(\frac{m\pi y}{b}\right) \qquad m \in \mathbb{Z}$$
$$Z\left(z\right) = \sinh\left(\gamma_{nm} z\right)$$

Where $\gamma_{nm} = \pi\sqrt{\left(\frac{n}{a}\right)^2 + \left(\frac{m}{b}\right)^2}$.

In order to satisfy the boundary condition at the top of the box, we look for a solution that is a linear combination of the solutions that we found:

$$\Phi\left(x, y, z\right) = \sum_{n,m} A_{nm} \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi y}{b}\right) \sinh\left(\gamma_{nm} z\right)$$

The rest of the problem is figuring out what the A_{nm} coefficients are.

We first apply the boundary condition $\Phi(x, y, c) = V(x, y)$:

$$\sum_{n,m} A_{nm} \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi y}{b}\right) \sinh(\gamma_{nm} c) = V(x, y)$$

This is a double Fourier series for $V(x, y)$, we want to find the coefficients of $V(x, y)$, and then match them to A_{nm}.

We use the orthogonality of the Fourier basis functions to determine the coefficients:

$$\sum_{n,m} A_{nm} \delta_{nn'} \delta_{m'm} \sinh(\gamma_{nm} c) = \int_0^a dx \, \frac{2}{a} \sin\left(\frac{n'\pi x}{a}\right) \int_0^b dy \, \frac{2}{b} \sin\left(\frac{m'\pi y}{b}\right) V(x, y)$$

$$A_{nm} = \frac{4}{ab} \frac{1}{\sinh(\gamma_{nm} c)} \int_0^a dx \int_0^b \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi y}{b}\right) V(x, y)$$

This is the furthest we can get without knowing $V(x, y)$.

Now let us consider a slightly harder problem.

Consider the same box, with the same coordinate system. We now impose the condition that $\Phi = 0$ on all sides of the box, with the catch that we have a point charge Q in the box, at some point (x_0, y_0, z_0).

We want to solve the equation:

$$\boldsymbol{\nabla}^2 \Phi = -\frac{Q}{\varepsilon_0} \delta(x - x_0) \, \delta(y - y_0) \, \delta(z - z_0)$$

Away from (x_0, y_0, z_0), we have Laplace's equation:

$$\boldsymbol{\nabla}^2 \Phi = 0$$

We now write solutions for $z > z_0$, and for $z < z_0$. We are allowed to do this, as long as we match the boundary cases. This is useful because we can reuse the solutions from the previous problem. We can write out the two cases:

$$\Phi(x, y, z < z_0) = \sum_{n,m} A^{<}_{nm} \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi y}{b}\right) \sinh(\gamma_{nm} z)$$

$$\Phi(x, y, z > z_0) = \sum_{n,m} A^{>}_{nm} \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi y}{b}\right) \sinh(\gamma_{nm} (c - z))$$

We determine the coefficients by matching at $z = z_0$.

We first enforce that the potential is continuous at the boundary $z = z_0$:

$$A^{<}_{nm} \sinh(\gamma_{nm} z_0) = A^{>}_{nm} \sinh(\gamma_{nm} (c - z_0))$$

The second boundary condition is determined by integrating the Poisson equation over an infinitesimal volume that includes the point (x_0, y_0, z_0). We choose the infinitesimal volume to be in the shape of a parallelopiped with sides Δx, Δy, and Δz, with $\Delta z \ll \Delta x, \Delta y$. Poisson's equation integrated over the parallelopiped then becomes:

$$\int_{V'} \boldsymbol{\nabla}^2 \Phi \, d^3x = -\frac{Q}{\varepsilon_0}$$

We can now apply Gauss's law to the parallelopiped:

$$\begin{aligned}\int_{V'} \boldsymbol{\nabla}^2\Phi \, d^3x &= \int_{S'} ds \; (\boldsymbol{\nabla}\Phi \cdot \hat{n}) \\ &= \int dx \, dy \; \left[\frac{\partial \Phi}{\partial z} \Big|_{z_0+\varepsilon} - \frac{\partial \Phi}{\partial z} \Big|_{z_0-\varepsilon}\right] \\ &= -\frac{Q}{\varepsilon_0}\end{aligned}$$

If we move the parallelopiped so that it does not contain the point charge:

$$\begin{aligned}\int_{V'} \boldsymbol{\nabla}^2\Phi \, d^3x &= \int_{S'} ds \, (\boldsymbol{\nabla}\Phi \cdot \hat{n}) \\ &= \int dx \, dy \frac{\partial \Phi}{\partial z} \Big|_{z_0+\varepsilon} - \frac{\partial \Phi}{\partial z} \Big|_{z_0-\varepsilon} \\ &= 0\end{aligned}$$

It follows from this that

$$\frac{\partial \Phi}{\partial z}\Big|_{z_0+\varepsilon} - \frac{\partial \Phi}{\partial z}\Big|_{z_0-\varepsilon} = -\frac{Q}{\varepsilon_0}\delta\left(x - x_0\right)\delta\left(y - y_0\right)$$

This gives us the next condition on the Fourier coefficients, substituting the expressions for Φ into the basis function expansion:

$$\sum_{n,m}\left[-A^{>}_{nm}\gamma_{nm}\cosh\left(\gamma_{nm}\left(c - z_0\right)\right) - A^{<}_{nm}\gamma_{nm}\cosh\left(\gamma_{nm}z_0\right)\right]\left[\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right)\right] = -\frac{Q}{\varepsilon_0}\delta\left(x - x_0\right)\delta\left(y - y_0\right)$$

Now using the same identity as before:

$$\frac{2}{L}\int_0^L dx \; \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n'\pi x}{L}\right) = \delta_{n,n'}$$

Multiplying both sides by a sine of n' term and a sin of m' term, then integrating, we find that

$$\gamma_{nm}\left[A^{>}_{nm}\cosh\left(\gamma_{nm}\left(c - z_0\right)\right) + A^{<}_{nm}\cosh\left(\gamma_{nm}z_0\right)\right] = \frac{4}{ab}\frac{Q}{\varepsilon_0}\sin\left(\frac{n\pi x_0}{a}\right)\sin\left(\frac{m\pi y_0}{b}\right)$$

Note that we know everything on the riht side, and we denote this as c_{nm}. We have two simultaneous equations for the A_{nm} coefficients, which if solved give us that:

$$\begin{aligned}A^{<}_{nm} &= \frac{c_{nm}\sinh\left(\gamma_{nm}\left(c - z_0\right)\right)}{\gamma_{nm}\sinh\left(\gamma_{nm}c\right)} \\ A^{>}_{nm} &= \frac{c_{nm}\sinh\left(\gamma_{nm}z_0\right)}{\gamma_{nm}\sinh\left(\gamma_{nm}c\right)}\end{aligned}$$

Now that we know what the A_{nm} coefficients are, we can substitute them into the definition of Φ. We are eventually left with

$$\Phi\left(x, y, z\right) = \sum_{n,m}\frac{4}{ab}\frac{Q}{\varepsilon_0}\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{n\pi x_0}{a}\right)\sin\left(\frac{m\pi y}{b}\right)\sin\left(\frac{m\pi y_0}{b}\right)\left[\frac{\sinh\left(\gamma_{nm}\left(c - z_{>}\right)\right)\sinh\left(\gamma_{nm}z_{<}\right)}{\gamma_{nm}\sinh\left(\gamma_{nm}c\right)}\right]$$

Where $z_{>}$ is the larger of the z and z_0, and $z_{<}$ is the smaller of the two. This enforces the correct symmetries of the system and matches the two definitions of Φ that we had when we partitioned the box into two sections.

3.11.2 Spherical Coordinates

Consider a sphere of radius a, with surface potential given by $V(a,\theta,\phi) = V_0(\theta,\phi)$. We want to find the potential outside the sphere. To do this, we need to solve Laplace's equation:

$$\boldsymbol{\nabla}^2 V = 0$$

with our given boundary conditions.

We can use separation of variables, and assume that our solution, denoted k, splits into 3 parts:

$$k(r,\theta,\phi) = R(r)\,P(\theta)\,\Phi(\phi)$$

Which can be inserted into Laplace's equation in spherical coordinates:

$$\frac{1}{r^2}\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{1}{P}\frac{1}{r^2\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = 0$$

Now we can rewrite this to get multiple equations that are equal to the same constant, as is usually done with separation of variables. The first equation we can separate out is the Φ equation:

$$-\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = m^2$$

Where we have chosen the constant of separation to be m^2. This has solution $\Phi = e^{im\phi}$, where m can be an integer (since $\Phi(0) = \Phi(2\pi)$).

The second equation we can separate out is the radial and angular portions:

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = -\frac{1}{P\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right) + \frac{m^2}{\sin^2\theta}$$

Which can also be separated out into two equations, with a different separation constant, chosen to be $l(l+1)$:

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = l(l+1)$$
$$\frac{1}{P\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right) + \frac{m^2}{\sin^2\theta} = l(l+1)$$

Looking at the radial equation, we have:

$$\frac{d^2R}{dr^2} + \frac{2}{r}\frac{dR}{dr} - l(l+1)\frac{R}{r^2} = 0$$

The solution to this will be some power law, since we have equal powers of r in the denominator:

$$R(r) = r^l$$
$$R(r) = \frac{1}{r^{l+1}}$$

Which we can find by assuming the solution is r^n for some n, and then plugging it back into the equation to find the conditions on n.

For the radial equation:

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right)+\left(l\left(l+1\right)-\frac{m^2}{\sin^2\theta}\right)P=0$$

If we make the substitution $x=\cos\theta$:

$$\frac{d}{dx}\left[\left(1-x^2\right)\frac{dP}{dx}\right]+\left[l\left(l+1\right)-\frac{m^2}{1-x^2}\right]P=0$$

This is a standard differential equation, it is the associated Legendre equation, and the solutions are the Legendre polynomials.

When θ goes from 0 to π, which is when we are covering the entire sphere with our boundary condition, l and m will actually be integers, and l will in fact be a positive integer, and m will range from $-l$ to l in integer increments (taking on a total of $2l+1$ values).

Let us solve this Legendre differential equation. To start, we consider the case where $m=0$. This corresponds to the case where our boundary conditions are independent of ϕ:

$$\frac{d}{dx}\left(\left(1-x^2\right)\frac{d}{dx}P_l\left(x\right)\right)+l\left(l+1\right)P_l\left(x\right)=0$$

The well-behaved solutions are the Legendre polynomials, which by convention are normalized such that $P_l\left(1\right)=1$. A compact representation of the polynomials is the Rodrigues formula:

$$P_l\left(x\right)=\frac{1}{2^l l!}\frac{d^l}{dx^l}\left(x^2-1\right)^l$$

The Legendre polynomials satisfy an orthogonality condition, and they form a complete basis on the range $[-1,1]$, much like sines and cosines in the Fourier series:

$$f\left(x\right)=\sum_{l=0}^{\infty}A_l P_l\left(x\right)$$

Where the coefficients can be determined via the orthogonality of the Legendre polynomials.

Now let us return to the case where $m\neq 0$. The solutions are now given by the associated Legendre functions. For positive m:

$$P_l^m\left(x\right)=\left(-1\right)^m\left(1-x^2\right)^{m/2}\frac{d^m}{dx^m}P_l\left(x\right)$$

The solutions for negative m are related to the positive m solutions, and therefore they are not linearly independent. The associated Legendre functions satisfy the orthogonality condition of the Legendre polynomials, and for every value of m, the $P_l^m\left(\cos\theta\right)e^{im\phi}$ functions form a complete set of orthogonal functions on the unit sphere. These are the spherical harmonics, and are denoted by $Y_{lm}\left(\theta,\phi\right)$:

$$Y_{lm}\left(\theta,\phi\right)=\sqrt{\frac{\left(2l+1\right)\left(l-m\right)!}{4\pi\left(l+m\right)!}}P_l^m\left(\cos\theta\right)e^{im\phi}$$

These have an orthogonality relation:

$$\int_0^{2\pi} d\phi \int_{-1}^{1} d(\cos\theta)\, Y_{l'm'} Y_{lm} = \delta_{l'l}\delta_{m'm}$$

This is the reason the prefactor is chosen, so that we are left with just a delta function.

From this discussion, it follows that the general form of the potential outside the sphere is given by

$$V(r,\theta,\phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left(a_{lm} r^l + b_{lm} \frac{1}{r^{l+1}} \right) Y_{lm}(\theta,\phi)$$

In this problem, we need the potential to go to 0 at infinity, so the a_{lm} terms must vanish, since r^l blows up at infinity. In the case where r becomes small (inside the sphere), we would have had $b_{lm} = 0$.

We now impose the boundary conditions at $r = a$:

$$V_0(\theta,\phi) = \sum_l \sum_m \frac{b_{lm}}{a^{l+1}} Y_{lm}(\theta,\phi)$$

Now finding the b_{lm} coefficients are the same as finding the coefficients of a Fourier series, we utilize the orthogonality condition. We multiply by $Y^*_{l'm'}$:

$$\int d\Omega\, Y^*_{l'm'}(\theta,\phi) \left[\sum_{lm} \frac{b_{lm}}{a^{l+1}} Y_{lm}(\theta,\phi) \right] = \int d\Omega\, Y^*_{l'm'}(\theta,\phi)\, V_0(\theta,\phi)$$

From this, we find that

$$b_{lm} = a^{l+1} \int d\Omega\, Y^*_{lm}(\theta,\phi)\, V_0(\theta,\phi)$$

Taking these and inserting them into the solution for the potential outside the sphere:

$$V(r,\theta,\phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left(\frac{a}{r} \right)^{l+1} Y_{lm}(\theta,\phi) \int d\Omega'\, Y^*_{lm}(\theta',\phi')\, V_0(\theta',\phi')$$

Let us do another example. We want to find the Dirichlet Green's function for the inside of a sphere of radius b centered at the origin, at some point inside the sphere $\boldsymbol{r}'$.

We have a Green's function that satisfies the condition:

$$\boldsymbol{\nabla}^2 G = -4\pi\delta^3(\boldsymbol{r} - \boldsymbol{r}')$$

subject to the condition that $G(\boldsymbol{r}, \boldsymbol{r}') = 0$ at $r = b$, which is the definition of Dirichlet boundary conditions.

We want to look for a solution that is expanded in terms of the spherical harmonics:

$$G(\boldsymbol{r}, \boldsymbol{r}') = \sum_{lm} g_{lm}(r)\, Y_{lm}(\theta,\phi)$$

We can insert this into the differential equation:

$$\sum_{lm}\left[\frac{d^2 g_{lm}}{dr^2}+\frac{2}{r}\frac{dg_{lm}}{dr}-l\left(l+1\right)\frac{g_{lm}}{r^2}\right]Y_{lm}\left(\theta,\phi\right)=-4\pi\delta^3\left(\boldsymbol{r}-\boldsymbol{r}'\right)$$

Multiplying both sides by $Y^*_{l'm'}\left(\theta,\phi\right)$ and integrating over the unit sphere:

$$\frac{d^2 g_{l'm'}}{dr^2}+\frac{2}{r}\frac{dg_{l'm'}}{dr}-l'\left(l'+1\right)\frac{g_{l'm'}}{r^2}=-4\pi\int d\Omega\, Y^*_{l'm'}\left(\theta,\phi\right)\delta^3\left(\boldsymbol{r}-\boldsymbol{r}'\right)$$

We can write out the delta function, $\delta^3\left(\boldsymbol{r}-\boldsymbol{r}'\right)=\delta\left(r-r'\right)\delta\left(\cos\theta-\cos\theta'\right)\delta\left(\phi-\phi'\right)/r^2$, and we note that the second two terms are 1 when we integrate over the unit circle, and thus we are left with:

$$\frac{d^2 g_{l'm'}}{dr^2}+\frac{2}{r}\frac{dg_{l'm'}}{dr}-l'\left(l'+1\right)\frac{g_{l'm'}}{r^2}=-\frac{4\pi}{r^2}Y^*_{lm}\left(\theta',\phi'\right)\delta\left(r-r'\right)$$

Thus we have an equation that the coefficients must satisfy:

$$\frac{d^2 g_{lm}}{dr^2}+\frac{2}{r}\frac{dg_{lm}}{dr}-\frac{l\left(l+1\right)}{r^2}g_{lm}=-\frac{4\pi}{r^2}Y^*_{lm}\left(\theta',\phi'\right)\delta\left(r-r'\right)$$

To solve this, we note that the right side is always zero, except when $r=r'$. Away from $r=r'$, we have that

$$g_{lm}\left(r\right)=A_{lm}r^l+B_{lm}\frac{1}{r^{l+1}}$$

In general, the values of A_{lm} and B_{lm} are different for $r>r'$ and $r<r'$. For $r<r'$, we have $A^<_{lm}$ and $B^<_{lm}$, and for $r>r'$, we have $A^>_{lm}$ and $B^>_{lm}$. We now have to stitch those two solutions together, across the point where the charge is.

The solution has to be smooth at $r=0$, so $B^<_{lm}=0$. If we now impose the Dirichlet boundary condition, $g_{lm}\left(r\right)=0$ for $r=b$:

$$\begin{aligned}A^>_{lm}b^l+B^>_{lm}\frac{1}{b^{l+1}}&=0\\ B^>_{lm}&=-A^>_{lm}b^{2l+1}\end{aligned}$$

We now only have two unique coefficients that we have to determine:

$$g_{lm}\left(r\right)=\begin{cases}A^<_{lm}r^l, & r<r'\\ A^>_{lm}\left(r^l-\frac{b^{2l+1}}{r^{l+1}}\right), & r>r'\end{cases}$$

Now we need to stitch these two solutions together at $r=r'$. Requiring continuity gives us that

$$\begin{aligned}g^<_{lm}\left(r'\right)&=g^>_{lm}\left(r'\right)\\ A^<_{lm}&=A^>_{lm}\left(1-\frac{b^{2l+1}}{\left(r'\right)^{2l+1}}\right)\end{aligned}$$

We will find the second condition by integrating our differential equation in a small region that contains $r=r'$. We do this because we expect something on the left side of the equation to create a

delta function. The only way for this to happen is for $\frac{dg_{lm}}{dr}$ to be discontinuous, and then the second derivative term generates a delta function. To make this rigorous, we integrate over a region from $r' - \varepsilon$ and $r' + \varepsilon$:

$$\int_{r'-\varepsilon}^{r'+\varepsilon} dr \left(\frac{d^2 g_{lm}}{dr^2} + \frac{2}{r}\frac{dg_{lm}}{dr} - \frac{l\left(l+1\right)}{r^2} g_{lm} \right) = -4\pi \int_{r'-\varepsilon}^{r'+\varepsilon} dr \, \frac{1}{r^2} Y_{lm}^* \left(\theta', \phi'\right) \delta\left(r - r'\right)$$

$$\frac{dg_{lm}}{dr} \Big|_{r+\varepsilon} - \frac{dg_{lm}}{dr} \Big|_{r'-\varepsilon} = -\frac{4\pi}{r'^2} Y_{lm}^* \left(\theta', \phi'\right)$$

The left side is found by the fact that g_{lm} is continuous over the region, so that term over a tiny region is zero. The first derivative has a discontinuity, but other than that it is also continuous, so the term vanishes. Finally, the second derivative term remains, since it is singular. We then use the fundamental theorem of calculus to get the evaluation at the bounds. This gives us the jump condition at $r = r'$, if we insert our expressions for g_{lm}:

$$A_{lm}^{>} \left(l r'^{(l-1)} + \left(l+1\right) \frac{b^{2l+1}}{r'^{(l+2)}} \right) - A_{lm}^{<} \left(l r'^{i(l-1)} \right) = -\frac{4\pi}{r'^2} Y_{lm}^* \left(\theta', \phi'\right)$$

Now we have two simultaneous equations for the A_{lm} coefficients.

Now solving for the coefficients, and substituting them back into the definition of g_{lm}, and substituting that into the definition of $G\left(\boldsymbol{r}, \boldsymbol{r}'\right)$:

$$G\left(\boldsymbol{r}, \boldsymbol{r}'\right) = \begin{cases} \sum_{lm} \frac{4\pi}{2l+1} r^l r'^l \left[\frac{1}{r'^{(2l+1)}} - \frac{1}{b^{2l+1}} \right] Y_{lm}^* \left(\theta', \phi'\right) Y_{lm} \left(\theta, \phi\right), & r < r' \\ \sum_{lm} \frac{4\pi}{2l+1} r^l r'^l \left[\frac{1}{r^{(2l+1)}} - \frac{1}{b^{2l+1}} \right] Y_{lm}^* \left(\theta', \phi'\right) Y_{lm} \left(\theta, \phi\right), & r > r' \end{cases}$$

Note that if we flip r and r', we see that the two equations switch with each other, along with the conditions. This will always be true when solving for the Green's function for Dirichlet boundary conditions.

The standard way of writing the Green's function is

$$G\left(\boldsymbol{r}, \boldsymbol{r}'\right) = \sum_{lm} \frac{4\pi}{2l+1} r_<^l r_>^l \left[\frac{1}{r_>^{2l+1}} - \frac{1}{b^{2l+1}} \right] Y_{lm}^* \left(\theta', \phi'\right) Y_{lm} \left(\theta, \phi\right)$$

Where $r_>$ is the larger of r and r', and $r_<$ is the smaller of the two.

In the limit where $b \to \infty$, the radius goes to infinity, we expect this to result in the point charge solution, since all we are saying is that the potential goes to zero at infinity.

$$\lim_{b\to\infty} G\left(\boldsymbol{r}, \boldsymbol{r}'\right) = \sum_{lm} \frac{4\pi}{2l+1} r_<^l r^{-(l+1)} Y_{lm}^* \left(\theta', \phi'\right) Y_{lm} \left(\theta, \phi\right)$$

This should be the Green's function for empty space:

$$G\left(\boldsymbol{r}, \boldsymbol{r}'\right) = \frac{1}{\left|\boldsymbol{r} - \boldsymbol{r}'\right|}$$

This turns out to be true, but requires many identities about Legendre polynomials to prove.

3.11.3 Cylindrical Coordinates

Let us deal with cylindrical coordinates with an example.

Consider an infinitely long cylinder of radius b, aligned so its axis lies on the z axis. The potential is independent of z, and is given by $V_0(\phi)$. We want to find the potential everywhere inside the cylinder.

We want to solve Laplace's equation:

$$\boldsymbol{\nabla}^2 V = 0$$

With the boundary conditions $V(\rho, \phi) = V_0$ at $\rho = b$.

In cylindrical, Laplace's equation takes the form:

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial V}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

We can immediately drop the z term, since we have no z dependence:

$$\begin{aligned}\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial V}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial \phi^2} &= 0 \\ \frac{\partial^2 V}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial V}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial \phi^2} &= 0\end{aligned}$$

Now by separation of variables, we aim to find a solution of the form $V(\rho, \phi) = R(\rho)\,\Phi(\phi)$. Inserting this into our equation:

$$\begin{aligned}\frac{\partial^2 R}{\partial \rho^2}\frac{1}{\Phi} + \frac{1}{\rho}\Phi\frac{\partial R}{\partial \rho} + \frac{1}{\rho^2}R\frac{\partial^2 \Phi}{\partial \phi^2} &= 0 \\ \frac{\rho^2}{R}\left[\frac{d^2 R}{d\rho^2} + \frac{1}{R}\frac{dR}{d\rho}\right] &= -\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2}\end{aligned}$$

This separates into two equations:

$$\begin{aligned}\frac{\rho^2}{R}\left[\frac{d^2 R}{d\rho^2} + \frac{1}{R}\frac{dR}{d\rho}\right] &= \nu^2 \\ -\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} &= \nu^2\end{aligned}$$

This second equation can be rewritten:

$$\frac{d^2\Phi}{d\phi^2} + \nu^2\Phi = 0$$

Which has solutions that are sines and cosines:

$$\begin{aligned}\Phi(\phi) &= A\sin(\nu\phi) + B\cos(\nu\phi) \\ &= A'\sin(\nu\phi + \alpha)\end{aligned}$$

Since $\Phi(\phi) = \Phi(\phi + 2\pi)$, this gives us the condition that $\nu \in \mathbb{Z}$.

Now let us consider the other equation:

$$\frac{d^2R}{d\rho^2} + \frac{1}{\rho}\frac{dR}{d\rho} - \frac{\nu^2}{\rho^2} = 0$$

Once again seeing that each term has the same powers of R over the same powers of ρ, we know that $R \sim \rho^n$. Inserting this, we find that $R = \rho^\nu$ and $R = \rho^{-\nu}$.

The general solution is then

$$V(\rho,\phi) = a_0 + b_0 \ln\rho + \sum_{n=1}^{\infty} a_n \rho^n \sin(n\phi + \alpha_n) + \sum_{n=1}^{\infty} \frac{b_n}{\rho^n} \sin(n\phi + \beta_n)$$

Since we care about the inside of the cylinder, we must include the case where $r = 0$, and thus we cna throw away the b_n and b_0 term, since the log and the ρ^{-n} explode there. Thus we are left with

$$V(\rho,\phi) = a_0 + \sum_{n=1}^{\infty} a_n \rho^n \sin(n\phi + \alpha_n)$$

We rewrite this as

$$V(\rho,\phi) = \frac{1}{2}c_0 + \sum_{n=1}^{\infty} \rho^n \left[c_n \cos(n\phi) + d_n \sin(n\phi)\right]$$

At $\rho = b$, we impose the boundary condition:

$$V(b,\phi) = V_0(\phi)$$
$$\frac{1}{2}c_0 + \sum_{n=1}^{\infty} b^n \left(c_n \cos(n\phi) + d_n \sin(n\phi)\right) = V_0(\phi)$$

Now we note that this is just a Fourier series for V_0. We invert this to find the coefficients c_n and d_n:

$$c_n = \frac{1}{\pi}\frac{1}{b^n}\int_0^{2\pi} d\phi' V_0(\phi') \cos(n\phi')$$
$$d_n = \frac{1}{\pi}\frac{1}{b^n}\int_0^{2\pi} d\phi' V_0(\phi') \sin(n\phi')$$

We can start putting these all together, by writing out the potential:

$$\begin{aligned}
V(\rho,\phi) &= \frac{1}{2\pi}\int_0^{2\pi} d\phi' V_0(\phi') + \sum_{n=1}^{\infty} \frac{1}{\pi}\frac{\rho^n}{b^n}\int_0^{2\pi} d\phi' V_0(\phi') \left[\cos(n\phi')\cos(n\phi) + \sin(n\phi')\sin(n\phi)\right] \\
&= \frac{1}{2\pi}\int_0^{2\pi} d\phi' V_0(\phi') + \sum_{n=1}^{\infty} \frac{1}{\pi}\frac{\rho^n}{b^n}\int_0^{2\pi} d\phi' V_0(\phi') \cos\left[n(\phi - \phi')\right] \\
&= \frac{1}{2\pi}\int_0^{2\pi} d\phi' V_0(\phi') \left[1 + 2\sum_{n=1}^{\infty} \left(\frac{\rho}{b}\right)^n \cos\left[n(\phi - \phi')\right]\right] \\
&= \frac{1}{2\pi}\int_0^{2\pi} d\phi' V_0(\phi') \left[1 + 2\sum_{n=1}^{\infty} \left(\frac{\rho}{b}\right)^n \mathrm{Re}\left[e^{in(\phi-\phi')}\right]\right]
\end{aligned}$$

Now consider:

$$\mathrm{Re}\left[1+2\sum_{n=1}^{\infty}\left(\frac{\rho}{b}\right)^{n}e^{in(\phi-\phi')}\right]=\mathrm{Re}\left[1+2\sum_{n=1}^{\infty}z^{n}\right]$$

Where $z=\frac{\rho}{b}e^{i(\phi-\phi')}$. This is a geometric series:

$$\mathrm{Re}\left[1+2\sum_{n=1}^{\infty}z^{n}\right]=\mathrm{Re}\left[1+\frac{2z}{1-z}\right]$$

For $|z|<1$. Thus we have that

$$\mathrm{Re}\left[1+2\sum_{n=1}^{\infty}\left(\frac{\rho}{b}\right)^{n}e^{in(\phi-\phi')}\right]=\frac{1-\frac{\rho^{2}}{b^{2}}}{1-\frac{2\rho}{b}\cos\left(\phi-\phi'\right)+\frac{\rho^{2}}{b^{2}}}$$

Inserting this into our expression for $V\left(\rho,\phi\right)$, we have that

$$V\left(\rho,\phi\right)=\frac{1}{2\pi}\int_{0}^{2\pi}d\phi'\,V_{0}\left(\phi'\right)\left[\frac{b^{2}-\rho^{2}}{b^{2}-2\rho b\cos\left(\phi-\phi'\right)+\rho^{2}}\right]$$

Which is our final solution for the potential inside the cylinder.