PHYS410 Notes (Fall 2022)

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1 Advanced Applications of Newtonian Mechanics

1.1 Newton's Second Law

Newtonian mechanics is centered around forces. His experimentally based understanding of the world generated a model about forces, but there are limits, which we can surpass if we look at things like Lagrangian mechanics. Newton's Second Law can be written in multiple different ways:

$$\boldsymbol{F}_{\text{net}} = \frac{d\boldsymbol{p}}{dt} \qquad \boldsymbol{F}_{\text{net}} = m\boldsymbol{a} \qquad \boldsymbol{F}_{\text{net}} = m\frac{d^2\boldsymbol{x}}{dt^2}$$

Perhaps the most general way of writing it would be

$$\ddot{\boldsymbol{x}} = \frac{1}{m}\boldsymbol{F}_{\text{net}}$$

What should we do to find the resulting motion if the force depends on nothing, if it is constant?

$$\boldsymbol{F} = \boldsymbol{C}$$

In this case, we have constant acceleration, where we can solve for the acceleration and then integrate twice to get $\boldsymbol{x}$. We get two arbitrary constants of integration from this, which can be determined from the initial conditions (or other conditions).

If the force depends on time, $\boldsymbol{F} = \boldsymbol{f}(t)$, we have that $\ddot{\boldsymbol{x}} = \frac{1}{m}\boldsymbol{f}(t)$. Once again, we integrate twice with respect to time, and once again have two arbitrary constants.

If the force depends on velocity, we have that $F = g(v)$. We can use the fact that $\dot{v} = \frac{1}{m}g(v)$, and we can rewrite this as $\frac{dv}{dt} = \frac{1}{m}g(v)$, which we can solve via separation of variables:

$$\int \frac{dv}{g(v)} = \frac{1}{m}t + A$$

From here, we have some function of v on the left, and we can then solve for $v(t)$, and then integrate once more, to get $x(t)$. This will also have two constants of integration.

If the force is a function of position, $F = h(x)$, we can make an educated guess for the form of the solution. Another method is to multiply both sides by $\dot{x}$:

$$\ddot{x} = \frac{h(x)}{m} \to \ddot{x}\dot{x}\frac{1}{m}h(x)\dot{x}$$

Now we integrate both sides:

$$\int \ddot{x}\dot{x}\,dx = \int \frac{1}{m}\dot{x}h(x)\,dt$$

Now using the fact that $\ddot{x} = d\dot{x}$ and $\frac{dx}{dt}dt = dx$, we are left with

$$\frac{1}{2}\dot{x}^2 = \frac{1}{m}\int h(x)\,dx + C$$

Let's do some problems. If the force is given by $F(t) = A\cos(\omega t)$, find $x(t)$. We can set up Newton's Second Law:

$$A\cos(\omega t) = m\ddot{x}$$

We can move the mass over:

$$\ddot{x} = \frac{A}{m}\cos(\omega t)$$

We can then integrate twice:

$$\dot{x} = \frac{A}{m\omega}\sin(\omega t) + C$$
$$x = -\frac{A}{m\omega^2}\cos(\omega t) + Ct + D$$

If we wanted to connect initial conditions, typically given as x_0 and v_0, we could plug in $t = 0$ into our solution, and then solve for the constants of integration ($x(0) = x_0$ and $\dot{x}(0) = v_0$). To do this for our problem:

$$x(0) = -\frac{A}{m\omega^2} + D = x_0 \to D = x_0 + \frac{A}{m\omega^2}$$

We can also take the derivative:

$$\dot{x} = \frac{A}{m\omega}\sin(\omega t) + C$$
$$\dot{x}(0) = C = v_0$$

This tells us that the final solution is given by

$$x(t) = -\frac{A}{m\omega^2}\cos(\omega t) + v_0 t + x_0 + \frac{A}{m\omega^2}$$

Another way to do this is to do definite integrals from the initial value to the current value.

We begin with Newton's Second Law:

$$m\frac{dv}{dt} = A\cos(\omega t)$$

Integrating:

$$\int m\frac{dv}{dt}\,dt = \int A\cos(\omega t)\,dt$$

Abusing notation a bit, we can cancel out the dt in the left integral:

$$\int_{v_0}^{v} m\,dv = \int_0^t A\cos(\omega t)\,dt$$

This integral gives us the result

$$mv\Big|_{v_0}^{v} = \frac{A}{\omega}\sin(\omega t)\Big|_0^t$$
$$mv - mv_0 = \frac{A}{\omega}\sin(\omega t) - 0$$
$$v(t) = v_0 + \frac{A}{m\omega}\sin(\omega t)$$

We can now do the second integral:

$$\int_{x_0}^{x} dx = \int_0^t \left[v_0 + \frac{A}{m\omega}\sin(\omega t)\right]\,dt$$
$$x - x_0 = \left[v_0 t - \frac{A}{m\omega^2}\cos(\omega t)\right] - \left[0 - \frac{A}{m\omega^2}\right]$$

Solving for x:

$$x(t) = x_0 + v_0 t - \frac{A}{m\omega^2}\cos(\omega t) + \frac{A}{m\omega^2}$$

Which is the same result we found using the constants of integration.

We can do another problem as well. Suppose we have a force that is a function of velocity, $F(v) = \frac{D}{v}$. We can write out Newton's Second Law:

$$m\frac{dv}{dt} = \frac{D}{v} \to mv\,dv = D\,dt$$

Where we have separated variables. We can then integrate both sides:

$$\int mv\,dv = \int D\,dt \to \frac{1}{2}mv^2 = Dt + C$$

We can solve this for $v(t)$, where we have implicitly redefined C:

$$v(t) = \sqrt{\frac{2Dt}{m} + C}$$

We could have also done this using the second approach:

$$\int_{v_0}^{v} v\,dv = \int_0^t \frac{D}{m}\,dt$$

Now let us do a problem where $F(x) = Cx$. We can write out Newton's Second Law:

$$m\ddot{x} = Cx$$

This diffeq is second order, linear, homogeneous. We assume a solution of the form $e^r t$, for some r. Note that the two conditions that allow us to do this are the fact that the equation is linear and homogeneous. Because of these two conditions, we can also leverage the property of superposition of solutions. We can insert our solution into the diffeq:

$$mr^2 e^{rt} = Ce^{rt}$$

We can then cancel out the e^{rt}s, because they can never be 0, so we are left with the quadratic:

$$mr^2 = C$$

Solving for r, we have that $r^2 = \frac{C}{m}$, giving the roots $r = \pm\sqrt{\frac{C}{m}}$. This gives us two solutions, which can be put together to get the general solution:

$$x(t) = Ae^{\sqrt{\frac{C}{m}}t} + Be^{-\sqrt{\frac{C}{m}}}$$

1.2 Drag

We have two regimes of drag that we will focus on. We have linear (viscous) grad, where $|F_d| = bv$, and quadratic (turbulent) drag, where $|F_b| = cv^2$. To get the sign right, the drag will always be opposite the direction of motion, $F_d = -bv$ in the linear case, and in the quadratic case we have either $-cv^2$ or cv^2, depending on the direction of v.

Let us talk about where these come from. If we have viscous drag, we have an object in a fluid, and the fluid has to move around the object, (pictorially they're called streamlines), the object warps the streamlines. The streamlines don't vary over time for viscous drag, i.e we have laminar flow. The drag force comes from the viscosity of the fluid, which is usually quantified by taking two parallel plates, each with area A. The top one moves with velocity v. The fluid in between the two plates will move, with fluid closer to the stationary plate moving slower than the speed at which fluid closer to the moving plate. This variation of the fluid speeds generates what's known as the gradient of fluid velocity, which is essentially $\frac{dv}{dy}$, where y represents the distance from the bottom plate. In this case, the drag force is given by

$$F_{\text{drag}} = A\frac{dv}{dy}\eta$$

Where through dimensional analysis we find that η has units of $\frac{N\cdot s}{m^2} = Pa \cdot s$, it is measured in Pascal seconds. This is also sometimes known as "poise".

If we go back to the geometry of an object in a fluid flow, we have a similar derivation of the drag force. In the case of a sphere:

$$F_{\text{drag}} = A_{\text{sphere}}\left(\frac{dv}{dy}\right)_{\text{average}}\eta = \pi D^2\left(\frac{3v}{D}\right)\eta = 3\pi Dv\eta$$

Where the area is the surface area of the sphere. This is known as Stokes' Law, and is restricted to a sphere. From this, we have our bv, where $b = 3\pi D\eta$ for a sphere.

For turbulent drag, we have a sort of handwavy argument. If we have an object moving through a fluid at some v, the drag force shold be proportional to the amount of fluid you have to move out of the way, which will be given by the density of the fluid times the cross-sectional area of the object, times the speed. We also need a factor of the speed to determine how fast we need to move the material:

$$F_{\text{drag}} \propto \rho A v \times v$$

For a sphere, $F_{\text{drag}} = \frac{1}{4}\rho A v^2 = \left(\frac{1}{2}C_d\right)\rho A v^2$, where C_d is known as the drag coefficient, and depends on the shape of the object.

1.2.1 2D Motion with Linear Drag

Let us go about solving the motion for linear drag. Let us for this calculation, assume that it is 1 dimensional motion, and upwards is positive. Since for linear drag the x and y components separate, in this case we can only look at the y case, knowing that the x case is independent.

Note that we cannot separate the quadratic drag case into 2 1D equations of motion.

We can look at the x component with the linear drag:

$$\dot{v}_x = \frac{F_d}{m} = -\frac{b}{m}v_x$$

We define $\tau = \frac{m}{b}$, which gives us that

$$\dot{v}_x = -\frac{1}{\tau}v_x$$

From this, we guess that $v_x = Ae^{-t/\tau}$. If we insert initial conditions, we find that $A = v_0$, and we see that as $t \to \infty$, we see that $v_x \to 0$, and $x \to C$, where C is some constant.

Now let us look at the y direction, where we have a more complicated diffeq. We have that

$$\dot{v}_y = -\frac{1}{\tau}v_y - g = -\frac{1}{\tau}(v_y + g\tau)$$

There are two ways to solve this. One way is to substitute to get rid of the constant term, where we define some $u(t) = v_y(t) + g\tau$. We then see that $\dot{u} = \dot{v}_y$, and thus a solution to the equation for u is a solution to v_y. We can then solve the differential equation in terms of u, and then convert back to $v_y(t)$. This would give us that

$$v_y(t) = (v_0 + g\tau)e^{-t/\tau} - g\tau$$

The second way to solve this is to separate variables and integrate:

$$\frac{dv_y}{v_y + g\tau} = -\frac{1}{\tau}\,dt$$

$$\int_{v_0}^{v} \frac{dv_y}{v_Y + g\tau} = \int_0^t -\frac{1}{\tau}\,dt$$

$$\ln(v_y + g\tau)\Big|_{v_0}^{v} = -\frac{t}{\tau}$$

Now we can exponentiate both sides:

$$\frac{v_y + g\tau}{v_0 + g\tau} = e^{-t/\tau}$$

Now we can solve for v_y:

$$v_y = (v_0 + g\tau)e^{-t/\tau} - g\tau$$

Which is the same result that we found using the other method.

We can now integrate this to get the result for $y(t)$:

$$y(t) = -(v_0\tau + g\tau^2)e^{-t/\tau} - g\tau t + C$$

Let us now look at the limiting cases. At $t = 0$, we see that $v_y(0) = v_0$, as expected. We would then choose C such that $y(0) = 0$. As $t \to \infty$, the velocity approaches $-g\tau$. This is the terminal velocity of the object:

$$v_{\text{terminal}} = -g\tau$$

Note that the negative here is just because of our coordinate system, normally we just define the terminal speed, which would be $g\tau$.

1.2.2 Falling Motion with Quadratic Drag

Our drag force vector can be given by $\boldsymbol{F}_d = -cv\boldsymbol{v}$, and the magnitude of the drag force will be cv_y^2. If we write out the differential equation:

$$\dot{v}_y = \frac{c}{m}v_y^2 - g$$

Note that we can already determine what the terminal velocity is, by setting the acceleration to 0, and we find that $v_{\text{ter}} = \sqrt{\frac{mg}{c}}$. We can rewrite our differential equation:

$$\dot{v}_y = -g\left(1 - \frac{v_y^2}{v_{\text{ter}}^2}\right)$$

We can again solve this via separation of variables:

$$\int \frac{dv_y}{1-\frac{v_y^2}{v_{\text{ter}}^2}} = -g\int dt$$

We can define $u = -\frac{v_y}{v_{\text{ter}}}$, which will make the separation of variables will have the setup:

$$v_{\text{ter}}^2 \int_{v_0} \frac{du}{1-u^2} = -g\int dt$$

This is a lookup integral, and it spits out the hyperbolic arctangent of u:

$$v_{\text{ter}}^2 \operatorname{arctanh}(u)\Big|_{v_0}^{v} = -gt$$

We end up with

$$y(t) = -\frac{v_{\text{ter}}^2}{g}\ln\left[\cosh\left(\frac{gt}{v_{\text{ter}}}\right)\right]$$

1.3 Forces on Moving Charged Particles

The force on a moving charged particle will be the electrostatic force plus the Lorentz force:

$$\boldsymbol{F} = q\boldsymbol{E} + q\boldsymbol{v}\times\boldsymbol{B}$$

To start with, we'll assume that $\boldsymbol{E} = 0$. The consequences of the cross product are that the direction of the force will be at right angles to the current velocity:

$$\boldsymbol{v}\cdot\dot{\boldsymbol{v}} = 0$$

This implies that the magnetic force doesn't change the speed of the particle, but does change the direction.

Let us assume that $\boldsymbol{B} = B\hat{z}$. We can now write Newton's Second Law:

$$m\dot{v} = -q\boldsymbol{v}\times\boldsymbol{B} = q(-v_xB\hat{y} + v_yB\hat{x})$$

$$m(\dot{v}_x\hat{x} + \dot{v}_y\hat{y} + \dot{v}_z\hat{z}) = qv_yB\hat{x} - qv_xB\hat{y}$$

From this, we can write this out component-wise:

$$m\dot{v}_x = qv_yB \quad m\dot{v}_y = -qv_xB \quad m\dot{v}_z = 0$$

Rewriting the two equations:

$$\dot{v}_x = \frac{qB}{m}v_y \quad \dot{v}_y = -\frac{qB}{m}v_x$$

These are coupled first order differential equations. There are different ways to solve this. Taylor's method is to define a complex variable $\eta = v_x + iv_y$. This is a "complex velocity". The advantage of this is that

$$\dot{\eta} = \dot{v}_x + i\dot{v}_y = \frac{qB}{m}\left[v_y - iv_x\right] = -i\frac{qB}{m}\left[v_x + iv_y\right] = -i\frac{qB}{m}\eta$$

We can define $\omega = \frac{qB}{m}$, and we then have that $\eta = \eta_0 e^{-i\omega t}$. This is a clockwise rotation in the complex plane, with some initial complex velocity η_0. From this solution, we can obtain v_x and v_y:

$$v_x(t) = \operatorname{Re}(\eta(t)) \quad v_y(t) = \operatorname{Im}(\eta(t))$$

This can be reduced to sines and cosines, as expected. We could then integrate this if we want to solve for the position in both directions.

Another way to solve the original equation is to plug one equation into the other, and generate a second order differential equation.

$$\dot{v}_x = \omega v_y \to \ddot{v}_x = \omega \dot{v}_y \to \ddot{v}_x = -\omega^2 v_x$$

This is something that we recognize as an oscillator diffeq, with angular frequency $\sqrt{\omega^2} = \omega$. This gets us the same result as the other method.

We can now ask what the radius of the circular motion will be:

$$R = \frac{|\eta_0|}{\omega} = \frac{mv}{qB}$$

1.4 Collisions

When talking about collisions, we have that the total momentum vector, is conserved in a collision (or really any interaction during which external forces are negligible). For a collision of two objects:

$$\boldsymbol{p}_{1i} + \boldsymbol{p}_{2i} = \boldsymbol{p}_{1f} + \boldsymbol{p}_{2f}$$

We can keep track of orthogonal components, and require them to be equal, before and after:

$$p_{1xi} + p_{2xi} = p_{1xf} + p_{2xf}$$

Or equivalently, $\Delta p_{1x} = -\Delta p_{2x}$. The same thing holds for all coordinate directions. The neat thing about this is that we can just compare before and after, the net change in momentum of each object. We know that $p = \gamma mv$, which is pretty much mv, as long as $v \ll c$. We can also think of vector equality graphically, where we are adding up the initial momentum vectors, and we need that sum to be the same as the sum of the final momentum vectors. When we mess with momentum triangles, we often use the law of cosines and sines:

$$c^2 = a^2 + b^2 - 2ab\cos\gamma$$

where γ is the angle opposite c. The law of sines is

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}$$

These rules mean that if you know any 3 pieces of information, you can determine the other 3. Its important to realize that we are dealing with momenta, not velocities. Because of this, we have to add/remove a factor of m. This works equally well for elastic and inelastic collisions, momentum conservation doesn't care.

1.5 Rocketry

Rocketry is an application of conservation of total momentum, where we have the rocket body and the fuel, which decreases in mass, or is separated from the rocket body.

We have to now make a distinction between a rocket and a turbofan. Rockets exhaust mass behind them to generate thrust, while turbofans burn fuel to spin fans that push against the ambient fluid medium, generally air.

If we take a soda bottle, and fill it with some liquid nitrogen, and put a stopper in the hole, it will build up pressure, and eventually eject the stopper, sending both the soda bottle and the stopper flying.

Let us look at a general rocket. The total mass of the rocket and its fuel, spent and unspent, is constant, m_0. Let us keep track of the rocket body plus its unspent fuel, $m(t)$. In a small time interval dt, while the engine is spitting out exhaust with speed v_e relative to the rocket body, the momentum at the beginning of the time interval is mv. The momentum after the time interval is $(m+dm)(v+dv) - dm(v-v_e)$. We can expand this equation, and we have that $m\,dv = -v_e\,dm$. This gives us the relation that $m\dot{v} = -v_e\dot{m}$, or $\dot{v} = -\frac{v_e}{m}\dot{m}$. If the thrust is constant, we can integrate, and we find that

$$v(t) = -v_e\left[\ln m - \ln m_0\right]$$

1.6 Angular Momentum

We will begin by thinking about a single particle, rather than a 3 dimensional rigid body. The angular momentum and rotation of a single particle is more straight-forward:

$$\boldsymbol{\ell} = \boldsymbol{r} \times \boldsymbol{p}$$

Where $\boldsymbol{p}$ is the linear momentum, $m\boldsymbol{v}$.

We can compute the derivative of ℓ:

$$\dot{\boldsymbol{\ell}} = \frac{d}{dt}(\boldsymbol{r} \times \boldsymbol{p}) = \dot{r} \times \boldsymbol{p} + \boldsymbol{r} \times \dot{\boldsymbol{p}}$$

Now noting that $\dot{\boldsymbol{r}} = \boldsymbol{v}$, which is colinear with $\boldsymbol{p}$, and thus we are left with

$$\dot{\boldsymbol{\ell}} = \boldsymbol{r} \times \dot{\boldsymbol{p}} = \boldsymbol{r} \times \boldsymbol{F}$$

This is known as the torque:

$$\dot{\boldsymbol{\ell}} = \boldsymbol{r} \times \boldsymbol{F} = \boldsymbol{\Gamma}$$

This is Newton's Second Law in angular form.

Let us do a quick review of computing cross products. Suppose we have two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, and the angle between them θ. Given these 3 characteristics, we can define the magnitude of the cross product as

$$|\boldsymbol{a} \times \boldsymbol{b}| = |\boldsymbol{a}||\boldsymbol{b}| \sin\theta$$

We can find the direction of the cross product via the right hand rule.

If instead we are given $\boldsymbol{a}$ and $\boldsymbol{b}$ written head to tail, and the projection of $\boldsymbol{a}$ in a direction perpendicular to $\boldsymbol{b}$, known as s. From this, we have that $|\boldsymbol{a} \times \boldsymbol{b}| = sb$, and we can once again use the right hand rule to compute the direction.

Suppose that we were given the vectors in some coordinate system:

$$\boldsymbol{a} = 2\hat{x} + 3\hat{y} + 5\hat{z} \quad \boldsymbol{b} = -\hat{x} - 5\hat{y} + 6\hat{z}$$

We can use the determinant method:

$$\boldsymbol{a} \times \boldsymbol{b} = \det \begin{pmatrix} \hat{x} & \hat{y} & \hat{z} \\ 2 & 3 & 5 \\ -1 & -5 & 6 \end{pmatrix}$$

Suppose we have a ball of mass m falling under gravity, with a position vector of $\boldsymbol{r}$. We can compute the angular momentum

$$\boldsymbol{\ell} = \boldsymbol{r} \times m\boldsymbol{v}$$

If we are given the distance from the origin in the x direction, known as s, we know that the magnitude will be $|\boldsymbol{\ell}| = m|\boldsymbol{v}|s$. Note that as the ball falls, the angular momentum will change, since it relies on the velocity. However, we note that if we move the origin to right under the ball, the angular momentum will be 0, and will remain 0 even as the ball falls, since the velocity and position vectors are colinear. Thus we see that in angular dynamics, the choice of origin leads to subtleties due to the cross products.

1.6.1 Kepler's Second Law

Kepler's Second Law states that the area swept out over some dt is the same for all points in an orbit. This is equivalent to a statement of conservation of momentum. We have some position vector in the orbit, $\boldsymbol{r}$, and some tangential velocity $\boldsymbol{v}$. In some amount of time dt, we sweep out a triangle, one side length being $\boldsymbol{r}$, and another being $\boldsymbol{v}\,dt$. We can use a property of triangles that states that the area is given by

$$A = \frac{1}{2}|\boldsymbol{a} \times \boldsymbol{b}|$$

We can use this:

$$A = \frac{1}{2}|\boldsymbol{r} \times \boldsymbol{v}\,dt| = \frac{1}{2m}|\boldsymbol{r} \times \boldsymbol{p}|\,dt$$

From this, we have that

$$\frac{dA}{dt} = \frac{1}{2m}|\boldsymbol{r} \times \boldsymbol{p}| = \frac{|\ell|}{2m} = C$$

Where C is some constant. This was first noticed empirically by Kepler, by looking through data.

1.6.2 Multiple Particles

For multiple particles, we define the total angular momentum to be the sum of the individual angular momenta:

$$\boldsymbol{L} = \sum_{\alpha} \boldsymbol{\ell}_{\alpha} = \sum_{\alpha} \boldsymbol{r}_{\alpha} \times \boldsymbol{p}_{\alpha}$$

We also note that the rate of change of the total angular momentum will be the total external torque:

$$\dot{\boldsymbol{L}} = \boldsymbol{\Gamma}_{ext}$$

We can write conservation of angular momentum as the claim that if $\boldsymbol{\Gamma}_{ext} = 0$, then $\boldsymbol{L}$ is constant, if the internal forces obey Newton's Third Law, and those forces are directed along the line connecting the two objects (For example, magnetic forces can violate the strong form of Newton's Third Law).

1.6.3 Moment of Inertia

For a fixed axis of rotation, we define the moment of inertia:

$$I = \sum_{\alpha} m_\alpha r_\alpha^2$$

Where r_α is the distance from particle α to the axis. For a continuous system, we can convert this to an integral:

$$I = \int r^2 \, dm$$

Where $dm = \rho dV$, turning the integral into

$$I = \int r^2 \rho \, dV$$

The moment of inertia can be used to find the angular momentum:

$$L_z = I\omega$$

Where ω is the angular velocity. We can take the derivative of this:

$$\dot{L}_z = I\dot{\omega} = \Gamma_z$$

We have some frictionless turntable, of mass M and radius R, initially at rest. We have an object of mass m with velocity v that hits the turn table b away from the center of the turntable, and sticks to the turntable. We want to know the final angular velocity of the table.

The initial angular momentum is $L_z = mvb$. After the collision, we have that $L_z = I\omega$. We now need to figure out what I is. This will be the sum of $I_{\text{turntable}}$ and I_{putty}:

$$I = mR + \frac{M}{2}R^2$$

Thus we have that

$$mvb = \left(mR + \frac{M}{2}R\right)\omega \to \omega = \frac{m}{\left(m + \frac{M}{2}\right)}\frac{vb}{R^2}$$

1.6.4 Center of Mass

We denote the center of mass vector as $\boldsymbol{R}_{cm}$ or $\boldsymbol{R}$, and it is given by the weighted sum of all the position vectors:

$$\boldsymbol{R} = \frac{1}{M}\sum_{\alpha} m_\alpha \boldsymbol{r}_\alpha$$

Where M is the total mass of the system. We can split this into components:

$$\boldsymbol{R}_x = \frac{1}{M}\sum_{\alpha} m_\alpha x_\alpha$$

and similarly for the y and z components. We can write Newton's Second Law for the entire system:

$$\boldsymbol{F}_{ext} = M\ddot{\boldsymbol{R}}_{cm}$$

The center of mass behaves like a single particle, and we can track its motion through space via the external forces on it. The reason that we can do this is because

$$M\ddot{\boldsymbol{R}} = \sum_\alpha m_\alpha \ddot{\boldsymbol{r}}_\alpha = \sum_\alpha \boldsymbol{F}_\alpha = \boldsymbol{F}_{ext}$$

Another special property is that the center of mass is that, if we choose the center of mass to be our origin, then $\dot{\boldsymbol{L}} = \boldsymbol{\Gamma}_{ext}$, all relative to the center of mass. This is a version of the angular form of the second law.

Let us do an example. We have a dumbbell, which is composed of two masses m a distance $2b$ apart from each other. We apply a force upwards on the left mass, $\boldsymbol{F}_{\text{mallet}}$ for a time Δt.

The change in total linear momentum is the impulse of the mallet, $F\Delta t$. The initial momentum was 0, so the final linear momentum is

$$\boldsymbol{p}_f = M\dot{\boldsymbol{R}} = \boldsymbol{F}\Delta t$$

Thus we have that

$$\dot{\boldsymbol{R}} = \frac{\boldsymbol{F}\Delta t}{2m}$$

We can similarly compute the change in total angular momentum:

$$|\boldsymbol{\Gamma}_{ext}| = Fb$$

And we know that $\boldsymbol{L}_i = 0$. Thus we have that

$$|\boldsymbol{L}_f| = I\omega = |\boldsymbol{\Gamma}_{ext}|\Delta t = Fb\Delta t$$

We can find that the moment of inertia of the dumbbell is $2mb^2$, and thus we have that

$$\omega = \frac{F\Delta t}{2mb}$$

We can then find the velocities of the left and right sides:

$$v_{left} = v_{cm} + \omega b = \frac{F\Delta t}{2m} + \frac{F\Delta t b}{2mb} = \frac{F\Delta t}{m}$$

Solving for the right velocity:

$$v_{right} = \frac{F\Delta t}{2m} - \frac{F\Delta t}{2m} = 0$$

1.7 Work-Energy Theorem

The work-energy theorem begins by defining the kinetic energy for a single particle:

$$KE = T = \frac{1}{2}m\boldsymbol{v}^2$$

The time derivative is

$$\frac{dT}{dt} = \frac{1}{2}m\frac{d}{dt}(\boldsymbol{v}\cdot\boldsymbol{v}) = \frac{1}{2}m\dot{\boldsymbol{v}}\cdot\boldsymbol{v}$$

Now noting that $m\dot{\boldsymbol{v}} = \boldsymbol{F}_{ext}$, we have that

$$\dot{T} = \boldsymbol{F}\cdot\boldsymbol{v} = \boldsymbol{F}\cdot\dot{\boldsymbol{r}}$$

We can then cancel out the dts, and we have that

$$dT = \boldsymbol{F} \cdot d\boldsymbol{r}$$

We define the right side to be the differential work, dW. If we have some path between two points, and some force law $\boldsymbol{F}$, we can partition the path into infinitely many $d\boldsymbol{r}$s, and add up all the pieces of differential work:

$$\int_a^b dT = \int_a^b \boldsymbol{F} \cdot d\boldsymbol{r}$$

The left side will just be the change in the kinetic energy, ΔT:

$$\Delta T = \int_a^b \boldsymbol{F} \cdot d\boldsymbol{r}$$

Let us do an example. Suppose the origin is the first point, and the second point is a quarter arc of a circle to the right of origin, to point $(1, 1)$. We can parameterize the path via a sweep angle from the point $(1, 0)$:

$$\boldsymbol{r} = (1 - \cos\theta, \sin\theta)$$

We can take a differential path segment:

$$d\boldsymbol{r} = (\sin\theta, \cos\theta)d\theta$$

We can specify the force law $\boldsymbol{F} = (y, 2x)$. We can then compute the integral:

$$W = \int_0^{\frac{\pi}{2}} (y\hat{x} + 2x\hat{y}) \cdot (\sin\theta\hat{x} + \cos\theta\hat{y})\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} y\sin\theta + 2x\cos\theta\, d\theta$$

Now using the fact that $x = 1 - \cos\theta$ and $y = \sin\theta$:

$$W = \int_0^{\frac{\pi}{2}} \sin^2\theta + 2(1 - \cos\theta)\cos\theta\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \sin^2\theta + 2\cos\theta - 2\cos^2\theta\, d\theta = 2 - \frac{\pi}{4}$$

This is the work done by the force on the particle as it moves along the path from points a to b.

We will now define a conservative force. If the work performed by a particular force on a particle travelling from position a to b only depends on the location of a and b and does not depend upon the path, the velocity, or the time, then the force is conservative.

The path independence will allow us to use Stokes' theorem:

$$W = \int_{\partial A} (\nabla \times \boldsymbol{F}) \cdot \hat{n}\, dA$$

We will use this to define a potential function.

1.8 Harmonic Oscillators

Oscillations occur when we have systems where forces that can go both ways, and are dependent on the displacement from an equilibrium point. These are restoring forces, it wants to return to equilibrium. Harmonic oscillations are those that have a restoring force that is proportional to the displacement from equilibrium. The byproducts of an oscillation being harmonic are that they have the same period, regardless of the amplitude of the oscillation.

Let us first talk about damping forces. The physical characteristics of a damping force include a proportionality to velocity. This generates a linear differential equation:

$$m\ddot{x} + b\dot{x} + kx = 0$$

Which we derived from the set up of

$$m\ddot{x} = -kx - b\dot{x}$$

Another property is that when we are oscillating, the pgase of the damping force is $\frac{\pi}{2}$ out of phase with the displacement. We now divide through by m, and define new varaibles, $\omega_0 = \sqrt{\frac{k}{m}}$:

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0$$

Where $\beta = \frac{b}{2m}$.

This is a linear equation, we we should have solutions of the form e^{rt}. We can insert this into the diffeq, and find the characteristic polynomial:

$$r^2 + 2\beta r + \omega_0^2 = 0 \to r = -\beta \pm \sqrt{\beta^2 - \omega_0^2}$$

1.8.1 Weakly Damped Systems

Any statement about a parameter with physical dimensions, being "small" or "large", has to be relative to some other quantity with the same units. In this case, weak means $\beta < \omega_0$. If this is true, then the actual oscillation frequency is shifted away from ω_0. We have that $r = -\beta \pm \sqrt{\beta^2 - \omega_0^2}$, and this is imaginary, so we can write it as

$$r = -\beta \pm i\sqrt{\omega_0^2 - \beta^2} \to -\beta \pm i\omega_u$$

Where we define $\omega_u = \sqrt{\omega_0^2 - \beta^2}$. We note that this frequency is $\leq \omega_0$. This is the actual oscillation frequency of the weakly damped system. We can also do this with the binomial approximation if $\beta \ll \omega_0$, which states that $(1+\epsilon)^a \approx 1 + a\epsilon$:

$$\omega_u = \omega_0\sqrt{1 - \frac{\beta^2}{\omega_0^2}} \to \omega_0 - \frac{\beta^2}{2\omega_0}$$

We have that $r = -\beta \pm i\omega_u$, so we can insert these back into the solution:

$$x = Ae^{-\beta t}e^{i\omega_u t} + Be^{-\beta t}e^{-i\omega_u t}$$

We see that both terms decay in ampltitude, at the same exponential rate. We can define a quality factor, Q:

$$Q = \frac{\omega_0}{2\beta} = \frac{\omega_0}{\gamma}$$

Q is dimensionless, and can be interpreted as how much the amplitude decays in Q oscillation cycles.

1.8.2 Overdamped Systems

If $\beta > \omega_0$, then both values of r are real, and we end up with no oscillation.

1.8.3 Driven Harmonic Oscillators

If we have a driving force, we write it using a complex exponential, but we imply the real portion of the expression:

$$m\ddot{x} + b\dot{x} + kx = F_d e^{i\omega t}$$

We then rewrite this equation

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = f_0 e^{i\omega t}$$

Where we note that $f_0 = \frac{F_d}{m}$. We can find the steady-state solution by guessing $x(t) = Ce^{i\omega t}$, and we can insert this into the diffeq and cancel out the exponential. We can then rewrite what we have left:

$$C[-\omega^2 + 2\beta\omega i + \omega_0^2] = f_0$$

We can then solve for C:

$$C = \frac{f_0}{(\omega_0^2 - \omega^2) + 2\beta\omega i}$$

We can find the amplitude and phase of this, A and δ:

$$x(t) = Ae^{i\delta}e^{i\omega t}$$

We can do this by taking the real part of C, and then rewriting what we find to find A and δ. We can find the real amplitude:

$$A = \frac{|f_0|}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}}$$

Or

$$A^2 = \frac{|f_0|^2}{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}$$

We can plot the frequency ω as a function of β, and we will see a resonance peak, centered at ω_0. As β decreases, we will see that the peak becomes taller and taller and more centered at ω_0, the driving frequency. We also note that larger Q defines a taller peak.

We have done the steady-state solution for the differential equation. However, any solution to the associated homogeneous diffeq can be added to the particular solution:

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0$$

From this, we have our general solution

$$x(t) = Ce^{i\omega t} + Ae^{-\beta t}\cos(\omega t) + Be^{-\beta t}\sin(\omega t)$$

2 Formulations of Mechanics

2.1 Lagrangian Mechanics

We have done mechanics through Newton's Laws so far, such as through the use of Newton's Second Law:

$$\frac{d\boldsymbol{p}}{dt} = \boldsymbol{F}_{net}$$

Where we focus on force, which is a vector, often in Cartesian coordinates, which allows us to separate the forces and vectors into components.

We have also taken advantage of energy conservation, where the total energy in the system is given by

$$E = \frac{1}{2}mv^2 + U(\boldsymbol{r})$$

This says nothing about force, but still gives us the necessary information about the system.

We will now move onto Lagrangian mechanics. This is based on Hamilton's Principle, which states that the path that a dynamical system actually takes from one point to another over a given time interval is the one that makes the action integral stationary. We define the action integral as S:

$$S = \int_{t_1}^{t_2} \mathcal{L}(x, \dot{x}, t)\, dt$$

This function $\mathcal{L}$ is known as the Lagrangian. In an ordinary intertial reference frame,

$$\mathcal{L} = K - U$$

The path of the system in Hamilton's principle is in this case $x(t)$. It could also involve $y(t)$ or $z(t)$, when talking about multidimensional motion. When we say "from one point to another" means that the starting and ending points are fixed, so we have t_1 and t_2, as well as $x(t_1)$ and $x(t_2)$.

This action integral is an example of a functional, which takes a function as input and calculates a value (whereas functions take values and output values).

Stationary means that small variations in the path do not change S to first order (in the size of the small variation). This means that path with this property is a minimum, maximum, or inflection point for S.

Let us change variables, and rewrite the action:

$$S = \int_{x_1}^{x_2} f(y, y', x)\, dx$$

Where we are not calling the function the Lagrangian just yet. We have the conditions that x_1, x_2, $y(x_1)$, and $y(x_2)$ are all fixed. We will be looking for a path $y(x)$ that makes the action stationary. If we consider adding some variation function $\eta(x)$ to the path, with some scale factor α, so we have a modified path

$$y_{mod}(x) = y(x) + \alpha\eta(x)$$

We now can find the derivative of the action with respect to α:

$$\frac{dS}{d\alpha} = \int_{x_1}^{x_2} \frac{df}{d\alpha}\, dx = \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y}\frac{\partial y}{\partial \alpha} + \frac{\partial f}{\partial y'}\frac{\partial y'}{\partial x} + \frac{\partial f}{\partial x}\frac{\partial x}{\partial \alpha} \right) dx$$

We note that $\frac{\partial y}{\partial \alpha} = \eta(x)$, and we note that $y'(x) = y_0'(x) + \alpha\eta'(x)$, so $\frac{\partial y'}{\partial \alpha} = \eta'(x)$. We also have that $\frac{\partial x}{\partial \alpha} = 0$. We can then rewrite our derivative:

$$\frac{dS}{d\alpha} = \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y}\eta x + \frac{\partial f}{\partial y'}\eta'(x) \right) dx = 0$$

We can use integration by parts on the right integral ($\int v\,du = uv - \int u\,dv$). We note that the endpoint term $uv\big|_{x_1}^{x_2}$ will be zero, because the variation is 0 at the endpoints of the path. Thus we are left with

$$\frac{dS}{d\alpha} = \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'} \right) \right] \eta(x)\,dx$$

We want this derivative to be zero for all $\eta(x)$. This requires that the terms in brackets must be zero:

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'} \right) = 0$$

This is the Euler-Lagrange equation, and will allow us to obtain the path $y(x)$ that the system takes. If there is more than one dimension, the path will have an Euler-Lagrange equation for each dimension, and the path would need to satisfy them simultaneously.

An example of this would be to find the shortest path between two points on a cylinder. For this, the action integral is really a calculation of the length of the path, $L = \int ds$.

Let us find the minimum path length between two points in the x y plane. We first want to integrate along a path to find the path's length. We first express the path length element ds in terms of dx and dy:

$$ds = \sqrt{dx^2 + dy^2}$$

If we have that the y coordinate is a function of x, then we can rewrite this path element:

$$ds = \sqrt{dx^2 + \left(\frac{dy}{dx}\right)^2 dx^2} = \sqrt{1 + y'^2}\,dx$$

And similarly for the case where x is a function of y:

$$ds = \sqrt{1 + x'^2}\,dy$$

We can also describe the path parametrically, with x and y as functions of t:

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{\left(\frac{dx}{dt}\right)^2 dt^2 + \left(\frac{dy}{dt}\right)^2 dt^2} = dt\sqrt{x'^2 + y'^2}$$

From this, we can write the total path integral as

$$L = \int_{t_0}^{t_1} \sqrt{\dot{x}^2 + \dot{y}^2}\,dt$$

Let us now use one of these methods. Let us suppose that the path is given by x as a function of y. For that reason, we will have an integral of the form

$$L = \int_{y_0}^{y_1} \sqrt{x'^2 + 1}\,dy$$

This is our action-like integral, as it is a function of x, x', and y. For the Euler-Lagrange equations, we have that

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial x'}\right) = 0$$

In this case, $f = \sqrt{1 + x'^2}$. We can see that the partial with respect to x is

$$\frac{\partial f}{\partial x} = 0$$

And we have that $\frac{\partial f}{\partial x'} = \frac{1}{2}(x'^2 + 1)^{-1/2} 2x'$. Inserting these into the Euler-Lagrange equation, we are left with:

$$(x'^2 + 1)^{-1/2} x' = C$$

Where C is a constant. We can then solve this, and we have that the path is a straight line, as we expect:

$$\frac{dx}{dy} = \sqrt{\frac{C^2}{1 - C^2}}$$

To find the actual line, we would insert the boundary conditions, the start and end points, which would determine C.

Suppose we now want to find the shortest path between two points on the surface of a cylinder. We can use spherical coordinates, (R, θ, z). Suppose we can describe our path with $\phi(z)$. We can write out the path element:

$$ds = \sqrt{sz^2 + (Rd\phi)^2} \to \sqrt{1 + R^2\phi'^2}\, dz$$

We can then write out the integral:

$$L = \int_{z_1}^{z_2} \sqrt{1 + R^2\phi'^2}\, dz$$

If we insert this into the Euler-Lagrange equations, we will find that the shortest path is givne by $\phi(z)$ increasing linearly with z.

Suppose we want to measure the elapsed time along the path, not the length of the path. The integrand should be an element of elapsed time. If we have the speed as some function $v(x, y)$, we can write out the time element:

$$dt = \frac{ds}{v(x, y)}$$

We can then write out the integral:

$$T = \int_{t_1}^{t_2} \frac{ds}{v(x, y)}$$

In a gravitational field, for motion that starts at point A and ends at point B, what shape ramp should we build so that sliding along the ramp, with no friction, minimizes the elapsed time to get to point B?

We generate our coordinate system with point A being at $(0, 0)$, and B at (x_2, y_2). We will assume that downwards is positive y, and we will assume that the path can be described by $x(y)$. We now need to find the speed at a given point along the path. From energy conservation, we know that the

the total energy is equal to 0 at point A, and $\frac{1}{2}mv^2 - mgy$ at point B. From this, we have that $v = \sqrt{2gy}$. The elapsed time is given by

$$T = \int_{path} dt = \int_{path} \frac{ds}{v} = \int_0^{y_2} \frac{dy\sqrt{x'^2+1}}{\sqrt{2gy}} = \frac{1}{\sqrt{2g}} \int_0^{y_2} \sqrt{\frac{x'^2+1}{y}}\, dy$$

We can now use the Euler-Lagrange equation, with our integrand being $f(x, x', y)$:

$$\frac{\partial f}{\partial x} - \frac{d}{dy}\left(\frac{\partial f}{\partial x'}\right) = 0$$

We can notice that $\frac{\partial f}{\partial x} = 0$, and that

$$\frac{\partial f}{\partial x'} = \frac{\frac{1}{2}(x'^2+1)^{-1/2}2x'}{\sqrt{y}}$$

Thus we have that

$$\frac{d}{dy}\left[\frac{\frac{1}{2}(x'^2+1)^{-1/2}2x'}{\sqrt{y}}\right] = 0$$

Thus we have that

$$\frac{(x'^2+1)^{-1/2}x'}{\sqrt{y}} = C$$

Where C is a constant. We can do some algebra:

$$2x' = C\sqrt{y}\sqrt{x'^2+1}$$

$$x'^2 = C^2 y(x'^2+1)$$

This gets us that

$$x'^2 = \frac{C^2 y}{1 - C^2 y}$$

We can square root this:

$$x' = C\sqrt{\frac{y}{1-C^2y}}$$

We can now separate variables and integrate:

$$\frac{dx}{dy} = C\sqrt{\frac{y}{1-C^2y}}$$

$$\int_0^{x_2} dx = C\int_0^{y_2} \sqrt{\frac{y}{1-C^2y}}\, dy$$

We will now redefine $a = \frac{1}{2c^2}$:

$$x_2 = \int_0^{y_2} \sqrt{\frac{y}{2a-y}}\, dy$$

This integral can be solved the substitution $y = a(1-\cos\theta)$, and then $dy = a\sin\theta\, d\theta$. We can then do some algebra and we find that

$$x_2 = \int_0^{\theta_f} \frac{1-\cos\theta}{\sin\theta} a\sin\theta\, d\theta = \int_0^{y_2} a(1-\cos\theta)\, d\theta$$

If we solve this, we find that the full path is described by $x = a(\theta - \sin\theta)$ and $y = a(1 - \cos\theta)$, where θ goes from 0 to θ_f. This describes a cycloid function. Thus the fastest path from A to B will be a section of the cycloid.

We have a block of mass m, on a wedge of mass M, both free to slide with zero friction. We want to figure out the dynamics of the system. We choose our coordinates to be w and b, where w is the edge of the wedge, and b is the distance of the small mass down the wedge.

We now need to write the kinetic and potential energies. We first note that $x_{block} = b\cos\alpha - w$, and $y_{block} = -b\sin\alpha$. We can then write out the kinetic energy:

$$K = \frac{1}{2}M\dot{w}^2 + \frac{1}{2}m\left[(-\dot{w} + \dot{b}\cos\alpha)^2 + (-\dot{b}\sin\alpha)^2\right] = \frac{1}{2}M\dot{w}^2 + \frac{1}{2}m\left[\dot{w}^2 - 2\dot{w}\dot{b}\cos\alpha + \dot{b}^2\right]$$

We have that the potential energy is given by $-mgb\sin\alpha$, and thus the Lagrangian is given by

$$\mathcal{L} = \frac{1}{2}M\dot{w}^2 + \frac{1}{2}m\left[\dot{w}^2 - 2\dot{w}\dot{b}\cos\alpha + \dot{b}^2\right] + mgb\sin\alpha$$

We can write out the Lagrange equations for w and b:

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{w}} = \frac{\partial\mathcal{L}}{\partial w}$$

$$\rightarrow \frac{d}{dt}\left[M\dot{w} + m\dot{w} - m\dot{b}\cos\alpha\right] = 0$$

From this, we know that $M\dot{w} + m\dot{w} - m\dot{b}\cos\alpha = k$, where k is some constant. The other way to look at it is to actually take the derivative:

$$(M+m)\ddot{w} - m\ddot{b}\cos\alpha = 0$$

We can do the EL equation for b, and we find that

$$\frac{d}{dt}\left[m\dot{b} - m\dot{w}\cos\alpha\right] = mg\sin\alpha$$

$$\rightarrow m\ddot{b} - m\ddot{w}\cos\alpha = mg\sin\alpha$$

$$\rightarrow \ddot{b} - \ddot{w}\cos\alpha = g\sin\alpha$$

We now have two equations, and we can combine them:

$$\ddot{b} - \frac{m\ddot{b}\cos\alpha}{M+m}\cos\alpha = g\sin\alpha$$

$$\ddot{b} = \frac{(M+m)g\sin\alpha}{(M+m) - m\cos^2\alpha} = \frac{(M+m)g\sin\alpha}{M + m\sin^2\alpha}$$

From this, we could integrate twice and then solve for b, and then use that to backsolve for w.

Suppose we have some fuzzy dice hanging from the mirror inside a car. There is some angle from the vertical, denoted ϕ. The car is accelerating with initial velocity v_0, and constant acceleration a. We want to calculate the kinetic energy for the dice. We have that

$$x = v_0 t + \frac{1}{2}at^2 - l\sin\phi \qquad y = -l\cos\phi$$

We can then write out the Lagrangian:

$$\mathcal{L} = \frac{1}{2}m\left[(v_0 + at - l\cos\phi\dot{\phi})^2 + l^2\sin^2\phi\dot{\phi}^2\right] + mgl\cos\phi$$

We note that this now explicitly depends on t. We can now write out the EL equation:

$$\frac{d}{dt}\left[\frac{1}{2}m\left(2(v_0 + at - l\cos\phi\dot{\phi})(-l\cos\phi) + 2(l\sin\phi\dot{\phi})l\sin\phi\right)\right] = \frac{\partial\mathcal{L}}{\partial\phi}$$

$$\to \frac{d}{dt}\left[v_0(-l\cos\phi) + at(-l\cos\phi) + l^2\cos^2\phi\dot{\phi} + l^2\sin^2\phi\dot{\phi}\right] = \frac{\partial\mathcal{L}}{\partial\phi}$$

$$\to \frac{d}{dt}\left[l^2\dot{\phi} - (v_0 + at)l\cos\phi\right] = \frac{\partial\mathcal{L}}{\partial\phi}$$

$$\to l^2\ddot{\phi} - al\cos\phi + (v_0 + at)l\sin\phi\dot{\phi} = \frac{\partial\mathcal{L}}{\partial\phi}$$

We can now do the right hand side:

$$l^2\ddot{\phi} - al\cos\phi + (v_0 + at)l\sin\phi\dot{\phi} = (v_0 + at)l\sin\phi\dot{\phi} - gl\sin\phi$$

We now simplify:

$$\ddot{\phi} = \frac{a}{l}\cos\phi - \frac{g}{l}\sin\phi$$

What does this tell us? We can try to find an equilibrium angle, when $\ddot{\phi} = 0$, and we find that $\tan\phi_e = \frac{a}{g}$.

We can also ask whether this is a stable or unstable equilibrium. There are two ways to do this. The first way to do so is to look at some small variation, $\phi = \phi_e + \epsilon$. We then compute $\ddot{\phi}$ for this varied point. We will then find that $\ddot{\epsilon} = -\left(\frac{a}{l}\sin\phi_e + \frac{g}{l}\cos\phi_e\right)\epsilon$. We see that this is a negative restoring force, proportional to ϵ. For a small deviation, this will be like a spring, and thus we have a stable equilibrium. The other method is to evaluate $\frac{d^2}{d\phi^2}$ at ϕ_e.

Suppose that we have a bead on a rotating wire loop, where the bead has coordinate θ from the rotation axis, and the loop is rotating with angular velocity Ω. The Lagrangian will be the sliding kinetic energy, $\frac{1}{2}m(R\dot{\theta})^2$, the rotational kinetic energy, $\frac{1}{2}m(R\sin\theta\Omega)^2$, and we have the potential energy, where we measure the height from the base, $U = mgR(1 - \cos\theta)$:

$$\mathcal{L} = \frac{1}{2}m(R\dot{\theta})^2 + \frac{1}{2}m(R\sin\theta\Omega)^2 - mgR(1 - \cos\theta)$$

We want to find the dynamics of θ:

$$\frac{d}{dt}\left[mR^2\dot{\theta}\right] = m(R\sin\theta\Omega)(R\cos\theta\Omega) - mgR\sin\theta\dot{\theta}$$

$$\to \ddot{\theta} = \sin\theta\cos\theta\Omega^2 - \frac{g}{R}\sin\theta$$

$$\ddot{\theta} = \left(\Omega^2\cos\theta - \frac{g}{R}\right)\sin\theta$$

This is the equation of motion for θ. We can find the equilibrium angle, which is where $\ddot{\theta} = 0$, so whenn $\sin\theta = 0$ or when $\Omega^2\cos\theta - \frac{g}{R} = 0$, or $\cos\theta = \frac{g}{R\Omega^2}$.

Let us have some general remarks on approximating. When we construct the Lagrangian, we could have a messy function of the coordinates. We generally have two options/approaches. One option is to apply the Euler-Lagrange equation to get the exact equation(s) of motion. We can then identify the equilibrium point or points, locations where the second derivative of the coordinate of interest is 0. We can then approximate the differential equation around the equilibrium point, and that should give us a differential equation in the form of a harmonic oscillator. This allows us to define an oscillation frequency ω.

The other approach is to approximate the Lagrangian, before we insert it into the Euler-Lagrange equations, for small deviations from the equilibrium point. From this, we should get a harmonic oscillator differential equation. For either case, we want to keep the lowest-order nonzero terms on both sides, and we note that the derivative of the coordinate will also be small, of roughly the same order as the coordinate itself. Note that when using the second method, we sometimes will need to approximate $\cos\theta$ as $1 - \frac{1}{2}\theta^2$.

2.1.1 Lagrange Multiplier method

Suppose we have a Lagrangian $\mathcal{L}(x, y, \dot{x}, \dot{y})$. A straightforward application of the Euler-Lagrange equation gives us two differential equations, one for x and one for y. These are in principle separate, and then we could impose constraints on those, such as using a constraint such as $x^2 + y^2 = L^2$ to substitute into the equations. Another way to do this is the Lagrange multiplier method. This method is based on expressing the constraint as $f(x, y) = C$, where C is a constant. We can add this to $\mathcal{L}$ without change the dynamics of the system, since it disappears when you take the derivatives. You add it along with a Lagrange multiplier, which is a function of time:

$$\mathcal{L} \to \mathcal{L} + \lambda f$$

Now the Euler-Lagrange equation becomes

$$\frac{d}{dt}\left[\frac{\partial \mathcal{L}_{new}}{\partial \dot{x}}\right] = \frac{\partial \mathcal{L}_{new}}{\partial x}$$

The added term has no dependence on $\dot{x}$, so the left side is the same as for the original $\mathcal{L}$:

$$\frac{d}{dt}\left[\frac{\partial \mathcal{L}}{\partial \dot{x}}\right] = \frac{\partial \mathcal{L}}{\partial x} + \lambda \frac{\partial f}{\partial x}$$

Now we solve simultaneously for $x(t)$, $y(t)$, and $\lambda(t)$, that satisfy the two Lagrange Equations. We now have 3 unknown functions, and 3 equations. The reason this can be useful, is because the flexibility of $\lambda(t)$ can help solve all 3 equations.

2.1.2 Conservation Laws

Suppose there is some coordinate q that the Lagrangian doesn't explicitly depend on (it can depend on $\dot{q}$). This is known as an "ignorable" coordinate. Then, from the E-L equation, we have that

$$\frac{\partial \mathcal{L}}{\partial q} = 0 \to \frac{d}{dt}\left[\frac{\partial \mathcal{L}}{\partial \dot{q}}\right] = 0$$

Thus we have that $\frac{\partial \mathcal{L}}{\partial \dot{q}}$ is a constant. This is the generalized momentum for q. This is a case of Noether's theorem:

Theorem 2.1. *If the dynamics of the system are invariant under a coordinate shift, then there is a conserved quantity.*

A trivial example of this is the free particle. The Lagrangian is

$$\mathcal{L} = \frac{1}{2}m\dot{x}^2$$

We have no explicit dependence on x. The generalized momentum of x is $m\dot{x}$, and is by Noether's theorem conserved, as we expected, linear momentum of the free particle is conserved.

For rotational motion, if there is no ϕ dependence in $\mathcal{L}$:

$$\mathcal{L} = \frac{1}{2}m(r\dot{\phi})^2 + \frac{1}{2}m\dot{r}^2 - U(r)$$

We see that ϕ is an "ignorable" variable. Via the property that we have just seen, we know that $\frac{\partial \mathcal{L}}{\partial \dot{\phi}}$ is conserved:

$$\frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \frac{1}{2}mr^2 2\dot{\phi} = mr^2\dot{\phi}$$

This is in fact, the angular momentum that we are familiar.

Let us look at another invariance example. Suppose $\mathcal{L}$ doesn't explicitly depend on t:

$$\mathcal{L}(q_1, \dot{q}_1, q_2, \dot{q}_2)$$

If we look at the total derivative with respect to time of this:

$$\frac{d\mathcal{L}}{dt} = \frac{\partial}{\partial q_1}\dot{q}_1 + \frac{\partial}{\partial q_2}\dot{q}_2 + \frac{\partial}{\partial \dot{q}_1}\ddot{q}_1 + \frac{\partial}{\partial \dot{q}_2}\ddot{q}_2$$

We now note that the first part of the first term on the right can be rewritten using the Euler-Lagrange equation as $\frac{d}{dt}\left[\frac{\partial \mathcal{L}}{\partial \dot{q}_1}\right]$. If we define this inner derivative as p_1, we can rewrite our total derivative as

$$\frac{d\mathcal{L}}{dt} = \frac{dp_1}{dt}\dot{q}_1 + p_1\frac{d}{dt}\dot{q}_1 + \frac{dp_2}{dt}\dot{q}_2 + p_2\frac{d}{dt}\dot{q}_2 = \frac{d}{dt}\left[p_1\dot{q}_1 + p_2\dot{q}_2\right]$$

From this, we have that

$$p_q\dot{q}_1 + p_2\dot{q}_2 - \mathcal{L} = C$$

Where C is constant. This combination is the formal definition of the Hamiltonian for a mechanical system.

2.1.3 Central Forces

Suppose we have two objects with a mutual force that is direct along the line connecting them, with a force magnitude that depends only on the separation distance:

$$\boldsymbol{F}(|\boldsymbol{r}_1 - \boldsymbol{r}_2|)$$

We define $\boldsymbol{r} = \boldsymbol{r}_1 - \boldsymbol{r}_2$, and we let $r = |\boldsymbol{r}|$. Thus we have that $\boldsymbol{F} = F(r)\hat{r}$. This began as a statement about 6 coordinates, the position coordinates of the two objects, but we have simplified it down to 1 coordinate. We can write out the Lagrangian:

$$\mathcal{L} = \frac{1}{2}m_1|\dot{\boldsymbol{r}}_1|^2 + \frac{1}{2}m_2|\dot{\boldsymbol{r}}_2|^2 - U(r)$$

If we change coordinates to center of mass coordinates:

$$\boldsymbol{r}_{cm} = \frac{m_1\boldsymbol{r}_1 + m_2\boldsymbol{r}_2}{m_1 + m_2}$$

Then we have that $\boldsymbol{r}_1 = \boldsymbol{r}_{cm} + \frac{m_2}{m_1+m_2}\boldsymbol{r}$, and likewise we have that $\boldsymbol{r}_2 = \boldsymbol{r}_{cm} - \frac{m_1}{m_1+m_2}\boldsymbol{r}$. We can now compute the derivatives:

$$\dot{\boldsymbol{r}}_1 = \dot{\boldsymbol{r}}_{cm} + \frac{m_2}{m_1+m_2}\dot{\boldsymbol{r}} \qquad \dot{\boldsymbol{r}}_2 = \dot{\boldsymbol{r}}_{cm} - \frac{m_1}{m_1+m_2}\dot{\boldsymbol{r}}$$

We can substitute these into the Lagrangian:

$$\mathcal{L} = \frac{1}{2}(m_1+m_2)|\dot{\boldsymbol{r}}_{cm}|^2 + \frac{1}{2}\left(\frac{m_1m_2}{m_1+m_2}\right)|\dot{\boldsymbol{r}}|^2 - U(r)$$

This has the form of a system with two non-interacting particles. The dynamics of $\boldsymbol{r}_{cm}$ and $\boldsymbol{r}$ are independent, the center of mass just acts like a free particle. We can just focus on the r dynamics:

$$\mathcal{L} = \frac{1}{2}\mu|\dot{\boldsymbol{r}}|^2 - U(r)$$

Where $\mu = \frac{m_1m_2}{m_1+m_2}$, is the reduced mass. Now we have 3 coordinates, the components of the $\boldsymbol{r}$ vector. From experience/intuition, we know that the motion will be in a plane. We can describe this with polar coordinates, r and ϕ. The Lagrangian we obtain is

$$\mathcal{L} = \frac{1}{2}\mu(\dot{r}^2 + (r\dot{\phi})^2) - U(r)$$

We note that this doesn't depend explicitly on ϕ, and thus $\frac{\partial\mathcal{L}}{\partial\dot{\phi}}$ is conserved. Thus we have that $\ell_z = \mu r^2\dot{\phi}$ is conserved.

We can write out the EL equation for r:

$$\frac{d}{dt}(\mu\dot{r}) = \mu r\dot{\phi}^2 - \frac{\partial U}{\partial r} \to \mu\ddot{r} = \mu r\dot{\phi}^2 - \frac{\partial U}{\partial r}$$

We can also substitute in the definition of the angular momentum, to get rid of $\dot{\phi}$:

$$\mu\ddot{r} = \frac{\ell^2}{\mu r^3} - \frac{\partial U}{\partial r}$$

This is 1D dynamics for r, with the "force" equal to $\mu\dot{r}^2 - \frac{\partial U}{\partial r}$. The additiional term, in terms of the angular momentum and the position is the centrifugal force. We can also write this as

$$\mu\ddot{r} = -\frac{\partial}{\partial r}\left[\frac{\ell_z^2}{\mu r^2} + U\right]$$

This combination term is the effective potential, it acts like a potential energy function, a function of the position. The extra term is sometimes call the "centrifugal barrier". This barrier term is always a positive contribution to the effective potential, while $U(r)$ can be positive or negative. Put another way, the added term is always repulsive, while $U(r)$ can be attractive or repulsive. If there is no dissipation, then something will be conserved. We can write that

$$K_r + U_{eff} = E$$

is conserved, where K_r is the radial kinetic energy, $\frac{1}{2}\mu\dot{r}^2$. This is all of the kinetic energy we have, since we have boiled the system down to a single coordinate, the radial component r. If we compare this to the true kinetic and potential energies in the 2D case:

$$\frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\phi}^2) + U(r) = E$$

In essence, we have kept the total energy the same, but "moved" the azimuthal kinetic energy to affect the potential term instead of the kinetic term. The dynamics of r will depend on E. At any r, we know that

$$|\dot{r}| = \sqrt{\frac{2}{\mu}(E - U_{eff}(r))}$$

If we look at U_{eff}, the local minima of this function will be the stable configurations at which we have circular motion. The actual orbit in the 2D plane depends on values of E and ℓ_z.

Let us now look at the stability of circular orbits. We can look at these like any equilibrium, but we use $U_{eff}(r)$. We have that if $\frac{d^2}{dr^2}U_{eff}(r) > 0$, then we have a stable solution, since the graph is concave upward. Let us consider central forces of the form $F = -kr^n$. The negative sign means that it is attractive. From this, we have that $U(r) = \frac{k}{n+1}r^{n+1}$, using the fact that $U = -\int F(r)\,dr$. The effective potential will then be given by

$$U_{eff}(r) = \frac{k}{n+1}r^{n+1} + \frac{\ell_z^2}{2\mu r^2}$$

If we have a circular orbit, then we have some r_0 stable point, which is when the effective potential is at a minimum:

$$\frac{dU_{eff}}{dr} = 0 \to kr_0^n - \frac{\ell_z^2}{\mu r_0^3} = 0 \to$$

We can then solve this for r_0, and we have that

$$r_0 = \left(\frac{\ell_z^2}{\mu k}\right)^{\frac{1}{n+3}}$$

We can check if it is stable:

$$\frac{d^2}{dr^2}U_{eff} = knr^{n-1} + \frac{3\ell_z^2}{\mu r^4} > 0$$

We can then solve this, and we have a stable case when

$$n > -3$$

We can now look at the frequency of small oscillations around r_0. We let $r = r_0 + \epsilon$, and $\ddot{r} = \ddot{\epsilon}$. We then insert this into our equation of motion, and we have that

$$\mu\ddot{\epsilon} = -k(r_0 + \epsilon)^n + \frac{\ell_z^2}{\mu}(r_0 + \epsilon)^{-3}$$

We divide by μ, and prep to use the binomial approximation:

$$\ddot{\epsilon} = -\frac{k}{\mu}r_0^n\left(1 + \frac{\epsilon}{r_0}\right)^n + \frac{\ell_z^2}{\mu^2}r_0^{-3}\left(1 + \frac{\epsilon}{r_0}\right)^{-3}$$

We can then use the binomial approximation, and then we are left with

$$\ddot{\epsilon} \approx -\frac{k}{\mu} r_0^n \left(1 + \frac{n\epsilon}{r_0}\right) + \frac{\ell_z^2}{\mu^2} r_0^{-3} \left(1 - \frac{3\epsilon}{r_0}\right)$$

We now use the fact that $r_0^{n+3} = \frac{\ell_z^2}{\mu k}$, to combine terms, and we are left with

$$\ddot{\epsilon} \approx -\left(\frac{\ell_z^2 (n+3)}{\mu^2 r_0^4}\right) \epsilon$$

For a circular or near circular orbit, we have that $\ell_z = \mu r^2 \omega_{orbit}$:

$$\ddot{\epsilon} \approx -((n+3)\omega_{orbit}^2)\epsilon$$

Thus we have that the angular frequency of small radial oscillations is

$$\omega = \sqrt{n+3}\omega_{orbit}$$

We have taken a 2D system and reduced it to an equivalent 1D problem with coordinate r. We get solutions for $r(t)$. There is also the motion of $\boldsymbol{r}_{cm}$, but it is motion with constant velocity. Putting it all together:

$$\boldsymbol{r}_1 = \boldsymbol{r}_{cm} + \frac{m_2}{m_1 m_2}\boldsymbol{r} \qquad \boldsymbol{r}_2 = \boldsymbol{r}_{cm} - \frac{m_1}{m_1 + m_2}\boldsymbol{r}$$

We have shown that the 2D Lagrangian had an azimuthal term in the kinetic energy:

$$\frac{1}{2}\mu(r\dot{\phi})^2$$

Which gets converted into the potential term in the 1D Lagrangian:

$$\frac{\ell_z^2}{2\mu r^2}$$

Which can be rewritten via $\ell_z = \mu r^2 \dot{\phi}$:

$$\frac{1}{2}\mu r^2 \dot{\phi}^2$$

We see that they are the same, but this is not just an algebraic regrouping of terms, since we **flipped the sign** of the term when looking at the second Lagrangian. The reason for this is based on the independent variables of the Lagrangians. These two are not the same Lagrangians. In the 2D case, we have the variables r and ϕ, our Lagrangian is of the form $\mathcal{L}(r, \phi)$. On the other hand, in the 1D Lagrangian, there is only one coordinate, r, and ℓ_z is fixed by construction. $\dot{\phi}$ is not fixed, because if r changes, $\dot{\phi}$ must change, because ℓ_z must be fixed. The difference in the sign comes from the fact that different things are considered fixed in both Lagrangians.

Assume that we have found an equilibrium point for the coordinate in a 1D system, such as r_0. We can heck stability and find the ω of small oscilations in two ways. The first method is to take the equation of motion from the Euler-Lagrange equation, and define $\epsilon = r - r_0$ and approximate for small ϵ. We can then look at the sign to see if it is a restoring force, and the value of the "spring constant" together with the mass to get ω:

$$m\ddot{\epsilon} = C\epsilon$$

If C is negative, then $\omega = \sqrt{\frac{|C|}{m}}$.

The other approach is more useful when we can characterize the effective potential. In this case, the "spring constant" is simply the second derivative of $U_{eff}(r)$, evaluated at r_0.

2.1.4 Orbits in Gravitational Potentials

Suppose we have two object in mutual orbit, with masses M and m, with a gravitational force of

$$F = \frac{Gmm}{r^2}$$

between them. Since the gravity force has the form $F = -kr^{-2}$, we already established that the period of radial variations is equal to the orbital period, $n = -2$, so $\omega = \omega_{\text{orbit}}$.

Let us now find the actual shape. From the gravitational potential energy, $U(r) = -\frac{GMm}{r}$, we get the equation of motion:

$$\mu\ddot{r} = -\frac{GMm}{r^2} + \frac{\ell^2}{\mu r^3}$$

We want to know $r(\phi)$, so we can make a polar plot. The trick here is to make a new variable, $u = \frac{1}{r}$, which allows us to compute the derivative of r:

$$\dot{r} = \frac{dr}{dt} = \frac{d}{dt}\frac{1}{u} = \frac{d}{d\phi}\frac{1}{u}\frac{d\phi}{dt} = -\frac{1}{u^2}\frac{du}{d\phi}\dot{\phi}$$

We know that $\ell = \mu r^2\dot{\phi}$:

$$\dot{r} = -\frac{1}{u^2}\frac{\ell_z u^2}{\mu}\frac{du}{d\phi} = -\frac{\ell_z}{\mu}\frac{du}{d\phi}$$

The next step is to compute the second derivative:

$$\ddot{r} = \frac{d}{dt}\dot{r} = \frac{-\ell_z}{\mu}\frac{d}{dt}\frac{du}{d\phi} = -\frac{\ell_z}{\mu}\frac{d}{d\phi}\frac{du}{d\phi}\frac{d\phi}{dt} = -\frac{\ell_z^2}{\mu^2}u^2\frac{d^2u}{d\phi^2}$$

We can now take our equation of motion and substitute in:

$$\mu\ddot{r} = -\frac{GMm}{r^2} + \frac{\ell^2}{\mu r^3} \to \mu\left(-\frac{\ell_z^2}{\mu^2}u^2\frac{d^2u}{d\phi^2}\right) = -GMmu^2 + \frac{\ell_z^2}{\mu}u^3$$

We can cancel out a u^2, divide by $\frac{\ell_z^2}{\mu}$, and we are left with

$$\frac{d^2u}{d\phi^2} = \frac{GMm\mu}{\ell_z^2} - u$$

This is almost a harmonic oscillator equation, but we are still alright since the extra term is a constant. We can let $v = u - \frac{GMm\mu}{\ell_z^2}$, and solve for v, and then backsolve for u:

$$u(\phi) = \frac{GMm\mu}{\ell_z^2} + A\cos(\phi - \phi_0)$$

Which can be written as

$$u(\phi) = B\left(1 + \epsilon\cos(\phi - \phi_0)\right)$$

Where $B = \frac{GMm\mu}{\ell_z^2}$ and $\epsilon = \frac{A}{B}$. We know that $r(\phi) = u^{-1}(\phi)$:

$$r(\phi) = \frac{C}{1 + \epsilon\cos(\phi - \phi_0)}$$

Where $C = \frac{\ell_z^2}{GMm\mu}$. We see that we have two parameters, ϵ and C.

If we plot this, the origin is one focus of the ellipse that is generated, and is the center of mass of the two body system. In the case of the Earth and the Sun, the center of mass point is almost exactly the same as the Sun's location, but in the case of two comparable masses, we have the two objects in elliptical orbits around the center of mass point, which is moving with constant velocity. The point on an orbit that is the furthest away from the focus is the apoapsis, (apogee if around the Earth, aphelion if around the Sun, apojove if around Jupiter, etc). And the point closest to the focus is the periapsis. The distance from the focus to the apoapsis is given by $\frac{C}{1-\epsilon}$, and the distance from the focus to the periapsis distance is given by $\frac{C}{1+\epsilon}$.

We have that $\dot{\phi}$ in the elliptical orbit is not constant, and we know that $\dot{\phi} = f(t)$:

$$\frac{d\phi}{dt} = f(t)$$

Integrating:

$$\int_0^{2\pi} d\phi = \int f(t)\, dt$$

In the case of Kepler's 3rd law, for a circular orbit:

$$\frac{mv^2}{r} = \frac{GMm}{r^2}$$

Which gets us that $v^2 = \frac{GM}{r}$. The speed is related to the period via $v = \frac{2\pi r}{T}$, so combining these things, we have that

$$\left(\frac{2\pi r}{T}\right)^2 = \frac{GM}{r} \to T^2 = \frac{4pi^2 r^3}{GM}$$

We can now replace r by a, the semi-major axis, for an elliptical orbit:

$$\frac{4\pi^2}{GM} a^3 = T^2$$

Note that this holds for $M \gg m$, which is useful for cases like the solar system.

2.2 Hamiltonian Mechanics

The Hamiltonian is defined using the Lagrangian. If we have $\mathcal{L}(q_1, q_2, \ldots, \dot{q}_1, \dot{q}_2, \ldots \dot{q}_n, t)$, the Hamiltonian is defined as

$$\mathcal{H} = \sum p_i \dot{q}_i - \mathcal{L}$$

Where $p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}}$, and is the generalized momentum for the q coordinate. This is deeper than just a variable substitution, for a couple reasons. The generalized momentum may not be trivial or familiar, and $\mathcal{H}$ is a function:

$$\mathcal{H}(q_1, q_2, \ldots q_n, p_1, p_2, \ldots p_n)$$

We see that we have the momenta, rather than the time derivatives. It is especially useful to focus on the momenta when they are conserved quantities, so p_i is constant, while $\dot{q}_i$ is not constant.

This set of coordinates, $\boldsymbol{q}$ and $\boldsymbol{p}$, are called phase space, versus the Lagrangian's state space. In phase space, the evolution of a system is based on the initial conditions, just like state space.

Notably, in phase space, trajectories do not intersect, if we start at two different points in phase space, and evolve the system, the paths will not intersect each other.

To obtain the Hamiltonian, we write down $\mathcal{L}$ in terms of q_i and $\dot{q}_i$, find all the generalized momenta, by taking derivatives with respect to $\dot{q}_i$, then compute the Hamiltonian. We can solve to get the $\dot{q}_i$ as a function of q_i, p_i, and possibly t, and then substitute to eliminate the $\dot{q}_i$.

Now we want to find how the system evolves in the Hamiltonian picture. Looking at derviatives of the Hamiltonian:

$$\frac{\partial \mathcal{H}}{\partial q} = \frac{\partial}{\partial q}\left[p\dot{q} - \mathcal{L}(q, \dot{q}, t)\right]$$
$$= p\frac{\partial \dot{q}}{\partial q} - \frac{\partial \mathcal{L}}{\partial q} - \frac{\partial \mathcal{L}}{\partial \dot{q}}\frac{\partial \dot{q}}{\partial q}$$

We see that the first and last term cancel, and thus we have that

$$\frac{\partial \mathcal{H}}{\partial q} = -\frac{\partial \mathcal{L}}{\partial q}$$

From the Euler-Lagrange equation, we know what the right side is equal to:

$$\boxed{\frac{\partial \mathcal{H}}{\partial q} = -\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) = -\dot{p}}$$

Looking at the other derivative of the Hamiltonian (skipping through the algebra, but we have a similar cancellation as above):

$$\boxed{\frac{\partial \mathcal{H}}{\partial p} = \dot{q}}$$

This pair of equations gives us the dynamics of the system. This is a set of two first-order differential equations. Perviously, we showed that if the Lagrangian has no explicit dependence on time, then $\mathcal{H}$ is conserved, and is the total energy. These two equations are called Hamilton's equations. We can think of plotting this in phase space, and these two give the evolution of the system in phase space, they define the trajectory.

Looking at a particle in free space, in 1D, with no force acting on it:

$$\mathcal{L} = \frac{1}{2}m\dot{x}^2$$

$$\frac{\partial \mathcal{L}}{\partial \dot{x}} = m\dot{x}$$

$$\mathcal{H} = m\dot{x}^2 - \frac{1}{2}m\dot{x}^2 = \frac{1}{2}m\dot{q}^2$$

Now using the fact that $\dot{x} = \frac{p}{m}$:

$$\mathcal{H} = \frac{p^2}{2m}$$

We can now take derivatives to get the motion:

$$\frac{\partial \mathcal{H}}{\partial p} = \dot{x} \to \dot{x} = \frac{p}{m} = \dot{x}$$

As expected, the velocity will remain the same, for a free particle.

Note that if the Lagrangian has no explicit dependence on t, then neither does the Hamiltonian. In that case, the Hamiltonian is a conserved quantity.

If the relationship between the generalized coordinates and inertial (Cartesian) coordinates is independent of time, then the generalized coordinates are said to be natural, and $\mathcal{H} = K + U$. This is because in natural coordinates, $\sum p_i \dot{q}_i = 2K$, and thus the Hamiltonian becomes $\mathcal{H} = 2K - \mathcal{L} = 2K - K + U = K + U$.

We also have ignorable coordinates, a generalized coordinate q_i that $\mathcal{L}$ does not depend on. We note that if $\mathcal{L}$ does not depend on q_i, then $\mathcal{H}$ will not depend on q_i either. Recall that this means that the generalized momentum associated with that coordinate was conserved/constant. In the Hamiltonian case, it includes p_i, but if thats a constant, its not really a variable, its a constant/parameter.

Let us look at a complex Atwood machine. This has a mass of $2m$ on the right side, and on the left we have a mass m attached to a spring, attached to another mass m. We assume that the pulley has negligible mass and rotational inertia. The spring has equilibrium length while hanging l_e.

We define our coordinates to be on the left side, where y is the distance from the pulley to the first mass, and we define the distance from the bottom of the first mass to the second mass to be $l_e + x$. We note that $k(l_e - l_0) = mg$, the displacement from equilibrium is given by the mass connected to the spring underneath. We can write out the kinetic energy:

$$K = \frac{1}{2}m\dot{y}^2 + \frac{1}{2}m(\dot{x} + \dot{y})^2 + m\dot{y}^2$$

Where the first term is the kinetic energy of the top hanging block, the second term is the kinetic energy of the spring and second mass (position is given by $y + l_e + x$, derivative is $\dot{x} + \dot{y}$), and the third term is the kinetic energy of the block on the right. Expanding this:

$$K = \frac{1}{2}m\dot{x}^2 + 2m\dot{y}^2 + m\dot{x}\dot{y}$$

The potential energy is given by

$$U = -mgy - mg(y + l_e + x) + 2mg(y - L) + \frac{1}{2}k(l_e + x - l_0)^2$$

Where L is the total length of the string. However, since constants in the potential energy will not affect the dynamics, we can replace this with just y. Likewise, we can drop the l_e term. Collecting terms:

$$U = -mgx + \frac{1}{2}k(x + (l_e + l_0))^2 = -mgx + \frac{1}{2}kx^2 + kx(l_e - l_0)$$

We can use the fact that $k(l_e - l_0) = mg$:

$$U = -mgx + \frac{1}{2}kx^2 + mgx = \frac{1}{2}kx^2$$

Putting all of these together, we have that

$$\mathcal{L} = \frac{1}{2}m\dot{x}^2 + 2m\dot{y}^2 + m\dot{x}\dot{y} - \frac{1}{2}kx^2$$

We can now calculate the momenta:

$$p_x = \frac{\partial \mathcal{L}}{\partial \dot{x}} = m\dot{x} + m\dot{y} = m(\dot{x} + \dot{y})$$

$$p_y = \frac{\partial \mathcal{L}}{\partial \dot{y}} = 4m\dot{y} + m\dot{x} = m(4\dot{y} + \dot{x})$$

Now inverting these relationships, to find $\dot{x}$ and $\dot{y}$ in terms of the momenta:

$$p_y - p_x = 3m\dot{y} \to \dot{y} = \frac{p_y - p_x}{3m}$$

$$\dot{x} = \frac{1}{m}\left(\frac{4}{3}p_x - \frac{1}{3}p_y\right)$$

Now let us calculate $\mathcal{H}$, in terms of the momenta. Since the coordinates are natural:

$$\mathcal{H} = K + U = \frac{1}{2}m\dot{x}^2 + 2m\dot{y}^2 + m\dot{x}\dot{y} + \frac{1}{2}kx^2$$

Substituting in the momenta:

$$\mathcal{H} = \frac{1}{2}m\left[3\dot{y}^2 + (\dot{y} + \dot{x})^2\right] + \frac{1}{2}kx^2 = \frac{1}{2m}\left[\frac{1}{3}(p_y - p_x)^2 + p_x^2\right] + \frac{1}{2}kx^2$$

We note that there is no dependence of y, and thus p_y is constant. We now use Hamilton's equations:

$$\dot{x} = \frac{\partial \mathcal{H}}{\partial p_x} = \frac{1}{2}m\left[-\frac{2}{3}(p_y - p_x) + 2p_x\right] = \frac{1}{3m}\left[4p_x - p_y\right]$$

$$\dot{y} = \frac{\partial \mathcal{H}}{\partial p_y} = \frac{1}{3m}(p_y - p_x)$$

Suppose we hold the block on the right still, and pull down the bottom one on the left, then let go. We will then have the initial conditions $x = x_0$, $y = y_0$, $\dot{x} = 0$, $\dot{y} = 0$. Initially, $p_x = p_y = 0$, from the definitions. We also know that p_y will stay 0. We thus have that $\dot{x} = \frac{4}{3m}p_x$, which means that $\ddot{x} = \frac{4}{3m}\dot{p}_x = -\frac{4k}{3m}x$. Thus we have that x oscillates with $\omega = \sqrt{\frac{4k}{3m}}$. Since $\dot{x}(0) = 0$, it must be a cosine:

$$x(t) = x_0 \cos \omega t \quad \text{with} \quad \omega = \sqrt{\frac{4k}{3m}}$$

2.3 Phase Space Orbits

Hamilton's equations tell how the system will evolve over time from wherever it current is in phase space. In a system with periodic motion, the curve in phase space will close, it will come back to where it was before. If the Hamiltonian does not have a time dependence, it will match onto the path, and repeat the "orbit".

When we have different orbits/trajectories in phase space, these represent different conserved $\mathcal{H}$s, which is sometimes the total energy.

If we have friction, or other non-conservative forces, we can have different paths that approach a certain path, such as an oscillating system that is losing energy, and forms a spiral.

The concept behind the next theorem, is roughly that if we designate a closed region of phase space and follow the evolution of all points inside that region, the "volume" occupied by them (as time elapses) is conserved. Notably, for chaotic systems, this is still true, but the shape can be highly distorted.

This relies on the divergence theorem:

$$\int_{\mathcal{S}} \boldsymbol{n} \cdot \boldsymbol{v}\, dA = \int_{\mathcal{V}} \nabla \cdot \boldsymbol{v}\, dV$$

Hamilton's equations can be shown to tell us that the region enclosed by a set of phase space trajectories has zero divergence:

$$\boldsymbol{v} = (\dot{q}_1, \dot{p}_1, \dot{q}_2, \dot{p}_2, \dots)$$
$$\nabla = \frac{\partial}{\partial q_1}\hat{q_1} + \frac{\partial}{\partial p_1}\hat{p}_1 + \dots$$

When we take the dot product in the right integral:

$$\nabla \cdot \boldsymbol{v} = \frac{\partial}{\partial q_1}\left(\frac{\partial \mathcal{H}}{\partial p_1}\right) + \frac{\partial}{\partial p_1}\left(-\frac{\partial \mathcal{H}}{\partial q_1}\right) + \dots$$

We can write this as

$$= \frac{\partial^2 \mathcal{H}}{\partial q_1 \partial p_1} - \frac{\partial^2 \mathcal{H}}{\partial p_1 \partial q_1} + \dots$$

These two terms cancel, and likewise for every following pair, thus we have 0.

When is Liouville's theorem valid? If the Hamiltonian is time dependent, then the divergence calculation would have to be modified, and we will end up at the last step with derivatives of a time dependent Hamiltonian, but the derivatives will still cancel out.

If we have nonconservative forces:

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}} = \frac{\partial \mathcal{L}}{\partial q} + F_{nc}$$

We can then modify Hamilton's equations, and we will find that this spoils the cancellation, if the acceleration due to the nonconservative force depends on p.

If the coordinates are not natural, then $\mathcal{H}$ is still conserved, it doesn't have to be the total energy.

Let us now look at the area of phase space orbits. If we have a harmonic oscillator, we have an ellipse as an orbit, and it has an area of

$$\pi\sqrt{\frac{2E}{k}}\sqrt{2mE} = 2\pi E\sqrt{\frac{m}{k}} = \frac{2\pi}{\omega}E = ET$$

Where T is the orbital period.

2.4 Noninertial Reference Frames

Let us think back to the fuzzy dice in the car example that we worked through. We fiund that there was an equilibrium angle, and $\tan\phi_e = \frac{a}{g}$. If we are riding in the car, then in your reference frame, the dice are hanging at (or maybe swinging around), this angle relative to the car's natural vertical. In the passenger reference frame, we have some gravitational force, and some "extra" force that is pushing the dice horizontally:

$$g_{eff} = \sqrt{g^2 + a^2}$$

Where a is the acceleration due to the extra force. The direction of "extra" or fictitious force, in the car's frame, is opposite the actual acceleration of the car in the inertial frame.

$$\boldsymbol{F}_{\text{moving}} = -m\boldsymbol{A}_{\text{frame}}$$

We really have a fictitious acceleration, we just infer a force because of Newtonian mechanics.

This component combines with gravity as being an equivalent thing.

2.4.1 Rotating reference frame

First, we need to describe how one reference frame is rotating relative to the inertial frame, specifying both the rotation axis and rate. We use the vector rotational velocity ω, with the right hand rule.

The linear speed is given by $v = r\omega$, and in fact we have that

$$\boldsymbol{v} = \boldsymbol{\omega} \times \boldsymbol{r}$$

The dynamics of the system come from using this derivative relationship. If something is fixed in the rotating frame, then

$$\frac{d\boldsymbol{r}}{dt} = \boldsymbol{\Omega} \times \boldsymbol{r}_{inertial}$$

But if $\boldsymbol{r}_{rot}$ is not constant, then

$$\boldsymbol{r}_{rot} = r_1\hat{e}_1 + r_2\hat{e}_2 + r_3\hat{e}_3$$

Then $\frac{d\boldsymbol{r}}{dt}_{rot}$ is given by

$$\frac{d\boldsymbol{r}}{dt}_{rot} = \frac{dr_1}{dt}\hat{e}_1 + \frac{dr_2}{dt}\hat{e}_2 + \frac{dr_3}{dt}\hat{e}_3$$

Seen in the inertial frame,

$$\boldsymbol{r}_{iner} = r_1(\hat{e}_1)_{iner} + r_2(\hat{e}_2)_{iner} + r_3(\hat{e}_3)_{iner}$$

Taking the derivative of this:

$$\left(\frac{d\boldsymbol{r}}{dt}\right)_{iner} = \frac{dr_1}{dt}(\hat{e}_1)_{iner} + r_1(\Omega \times (\hat{e}_1)_{iner}) + \dots$$

Thus we have that

$$\left(\frac{d\boldsymbol{r}}{dt}\right)_{iner} = \left(\frac{d\boldsymbol{r}}{dt}\right)_{rot} + \Omega \times \boldsymbol{r}_{iner}$$

In fact, this is true for any vector quantity. If we now look at the second derivative (where we are dealing with some arbitrary vector quantity $\boldsymbol{Q}$), we will find that

$$\frac{d^2\boldsymbol{r}}{dt^2}_{iner} = \frac{\boldsymbol{F}}{m} = \ddot{\boldsymbol{r}} + \dot{\boldsymbol{\Omega}} \times \boldsymbol{r} + 2\boldsymbol{\Omega} \times \dot{\boldsymbol{r}} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{r})$$

The first term is the Euler force, the second term is the Coriolis effect, and the third term is the centrifugal force.

The centrifugal force depends on position but not velocity, and is always radially outward from the the axis of rotation, not the origin. The magnitude is equal to $mr_\perp\Omega^2$.

The Coriolis force depends on the velocity, not the position, and has magnitude $2mv\Omega\sin\theta$, where θ is the angle between $\boldsymbol{v}$ and $\boldsymbol{\Omega}$. This should be maximal when they are perpendicular.

We can explain tidal forces via the interaction of the centrifugal forces as well as the force due to gravity of the moon.

Suppose we have a totally water covered planet, with slow rotation, and we want to find the shape it would have. The buoyancy force is perpendicular to the equilibrium surface, and must be cancelling the sum of the gravity and tidal forces. If we do this out, we would find that the planet would not

be spherical. Really, the water surface will follow an equipotential, the net potential energy will be constant over the whole surface. For the tidal force, this is approximately an ellipsoid.

For a flat rectangular container, filled with some water, we find a quadratic equipotential surface:

$$U = U_0 + \frac{1}{2}m\Omega^2 x^2$$

The Earth also hastides, the deformation of the solid Earth due to its rotation, but on the order of magnitude of centimeters (?). Over distance of a few kilometers, the distance can stretch by a fraction of a millimeter.

3 Elasticity

We know how springs work, we can define the force due to a spring via Hooke's Law, $F = -kx$. We can measure the spring constant by measuring the equilibrium length, and hanging a known mass from the spring. The gravitational force will balance the spring force, and thus we can solve for k.

If we connect springs in series, we find that the spring constant acts like resistors in parallel:

$$\frac{1}{k_{eff}} = \frac{1}{k_1} + \frac{1}{k_2}$$

In parallel, we have that the spring constant acts like resistors in series:

$$k_{eff} = k_1 + k_2$$

Wires and thin rods can act as a spring. The relevant property describing intrinsic elasticity is called the Young's Modulus, Y. This is defined as the stress over the strain. To check units, stress has units of pressure, and the strain is unitless, thus the Young's modulus has units of pressure, Newtons per square meter, or Pascals.

The stress is defined as the force divided by the cross-sectional area, $\frac{F}{A}$. Strain is defined as the change in length divided by the equilibrium length, $\frac{\Delta L}{L}$, also known as the fractional stretching. From these, we find that

$$Y = \frac{FL}{A\Delta L} \to F = \frac{YA\Delta L}{L} = \frac{YA}{L}\Delta L$$

We see that the spring constant in terms of the Young's Modulus, length, and cross-sectional area, is given by $\frac{YA}{L}$.

From this, we can intuitively see that placing them in parallel would increase the cross-sectional area, increase k, and placing them in series would increase L, decreasing k.

Let us now talk about elasticity in 3 dimensions. Suppose we have a cube of material, embedded in a larger block of solid material. The cube can compress or grow in all directions equally. This would be a "bulk" change in volume, and works for liquids as well as solids.

The cube can stretch or shrink in just one linear dimension, this is tension or compression. Young's Modulus applies when volume is conserved, and thus this is not the same as the streching of a wire, since the cross-sectional area decreases as the length increases in a stretching wire, which is not the same in this linear shift.

We have shear, or skew, as well as twists, or torsion.

Any elastic modulus involves a strain, a fractional change. For bulk modulus, that is $\frac{\Delta V}{V}$:

$$B = \frac{\Delta P}{-\frac{\Delta V}{V}}$$

For shear, the stress is $\frac{F}{A}$, where F is the sideways force, and A is the area of the plane that it is acting on, for example:

$$\frac{F_x}{A_{z \text{ view}}}$$

Volume and Surface Forces

We can represent the forces on a volume as a matrix σ, such that:

$$\boldsymbol{F} = \sum_i \sigma_{ij} dA_j$$

Where dA_j is the differential area vector in the j direction. We can write this as a tensor, which is a linear mapping of one vector to another vector in a geometrically consistent way. This is a rank 2 tensor:

$$\boldsymbol{F} = \boldsymbol{\Sigma}\, d\boldsymbol{A}$$

A proper stress tensor should be symmetric, for example:

$$\boldsymbol{\Sigma} = \begin{bmatrix} 2 & 3 & 0 \\ 3 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

This is symmetric. We want to find the force on a surface described by $x^2 + 2y + z = 7$ at $(2, 0, 3)$.

We begin by finding the normal vector, which is necessary to get the area vector. We can find this via the gradient of the function at the point given:

$$\nabla(x^2 + 2y + z) = \langle 2x, 2, 1 \rangle$$

Evaluating this at $(2, 0, 3)$ gets $\langle 4, 2, 1 \rangle$. We can normalize this:

$$\hat{n} = \frac{1}{\sqrt{21}} \langle 4, 2, 1 \rangle$$

The force is given by $\boldsymbol{F} = \boldsymbol{\Sigma} d\boldsymbol{A}$:

$$\boldsymbol{F} = \begin{bmatrix} 2 & 3 & 0 \\ 3 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \frac{1}{\sqrt{21}} \begin{bmatrix} 4 & 2 & 1 \end{bmatrix} dA = \frac{1}{\sqrt{21}} \begin{bmatrix} 14 & 8 & 1 \end{bmatrix} dA$$

We have a derivatives matrix, $\boldsymbol{D}$, such as

$$\boldsymbol{D} = \begin{bmatrix} .03 & .01 & .03 \\ .03 & .01 & 0 \\ -.03 & -.06 & .02 \end{bmatrix}$$

This matrix gives the displacement of any point.

We can remove any overall rotation, separating out the anti-symmetric part, which gives us the strain tensor, $\boldsymbol{E}$:

$$\boldsymbol{E} = \frac{1}{2}(\boldsymbol{D} + \boldsymbol{D}^T)$$

In this case, we have that

$$\boldsymbol{E} = \frac{1}{2}\left[\begin{bmatrix} .03 & .01 & .03 \\ .03 & .01 & 0 \\ -.03 & -.06 & .02 \end{bmatrix} + \begin{bmatrix} .03 & .03 & -.03 \\ .01 & .01 & -.06 \\ .03 & 0 & .02 \end{bmatrix}\right] = \begin{bmatrix} .03 & .02 & 0 \\ .02 & .01 & -.03 \\ 0 & -.03 & .02 \end{bmatrix}$$

This is the strain tensor, and is a symmetric matrix. The trace of $\boldsymbol{E}$ divided by 3 tells us the overall dilatation, e, also called spherical strain. The trace is given by the sum of the on-diagonal elements, in this case, $.03 + .01 + .02 = .06$, so $e = \frac{1}{3}.06 = .02$.

We can write the strain tensor as $\boldsymbol{E} = e\mathbf{1} + \boldsymbol{E}'$, which we can solve for $\boldsymbol{E}'$:

$$\boldsymbol{E}' = \boldsymbol{E} - e\mathbf{1} = \begin{bmatrix} .01 & .02 & 0 \\ .02 & -.01 & -.03 \\ 0 & -.03 & 0 \end{bmatrix}$$

We note that this matrix has 5 degrees of freedom, 6 independent values and then the constraint that it must be traceless.

If the elastic response of the material is linear and rotationally invariant, then the most general possible relation is

$$\boldsymbol{\Sigma} = \alpha e\mathbf{1} + \beta\boldsymbol{E}'$$

Compare this to the linear version, were we said that $\frac{F}{A} = Y\frac{\Delta L}{L}$. We that stress is equal to a scaled version of strain.

We can expand out $\boldsymbol{E}'$:

$$\boldsymbol{\Sigma} = \alpha e\mathbf{1} + \beta(\boldsymbol{E} - e\mathbf{1}) = (\alpha - \beta)e\mathbf{1} + \beta\boldsymbol{E}$$

An alternative set of parameters is the "Lamé parameters", where $\alpha - \beta$ is 3λ and β is 2μ.

We can solve for $\boldsymbol{E}$ now:

$$\boldsymbol{E} = \frac{1}{3\alpha\beta}\left[3\alpha\boldsymbol{\Sigma} - (\alpha - \beta)(\mathrm{tr}\boldsymbol{\Sigma})\mathbf{1}\right]$$

In some special caese, we can relate α and β to elastic moduli that we have already introduced. If we have the bulk modulus, where we have the effect of a pressure change with no shear:

$$\boldsymbol{\Sigma} = -p\mathbf{1}$$

And thus

$$\boldsymbol{E} = \frac{p}{\alpha\beta}\left[-\beta\right]\mathbf{1} = -\frac{p}{\alpha}\mathbf{1}$$

From the definition of the bulk modulus, we have that

$$BM = \frac{\alpha}{3}$$

We can also define the shear modulus in the presence of no volume change:

$$SM = \frac{\beta}{2}$$

4 Solid Body Rotation

There was a period where I didn't really take notes, these are the lecture notes starting at 11/29/22.

Homework 11 Question 5 asks you to analyze a sliding cube that hits a low step, causing it to tip. When the cube hits the lip, the lip applies force to suddenly stop that corner of the cube from sliding. When that happens, are any of our 3 quantities conserved? Kinetic energy is not conserved, since there is a force over a small distance We can think of the cube or the lip being slightly deformed as the cube comes to a stop, so we have work done by the contact force. The linear momentum is not conserved, since we have an impulse acting on the cube.

The angular momentum is conserved. There are two possible origins that we could use. If we choose the lip corner as the origin, then the sliding cube has nonzero angular momentum to begin with. This will be given by $\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p}$, or $r_\perp p$. At the time of collision, the force from the lip on the cube applies 0 torque, since it is acting at the origin. Thus the angular momentum will not change with this as our origin.

The other choice of origin would be the center of mass of the cube. If so, the sliding cube has 0 angular momentum initially. In this case, the force from the lip does apply a torque to the cube, and that starts to make it rotate around the origin. If we pick this as the origin, we see that it is not conserved.

Thus we see that the conservation of the angular momentum depends on the choice of origin.

To find the total kinetic energy of a system that is rotating and has a moving center of mass, when should you include a translation kinetic energy for the center of mass motion, and when should you not use one?

Suppose we have a uniform rod, an we imagine that it is rotating around an axis at the end, and its center of mass is at the center of the rod. If we have arbitrary motion of the center of mass, the safe way to write out the kinetic energy will be

$$K_{tot} = K_{\text{CM Translational}} + K_{\text{Rotational about CM}}$$

However, oftentimes (as in our rod case), the object is not rotating about the center of mass. If the CM is travellling along a circular path, you can calculate the moment of inertia around the actual axis:

$$K_{tot} = K_{\text{Rotational about Axis}} = \frac{1}{2} I_{\text{axis}} \omega^2$$

These two are equivalent ways of writing out the kinetic energy, but we need to remember that $I_{CM} \neq I_{\text{axis}}$, although they can be related by the parallel axis theorem.

We previously computed the inertia tensor for a rectangular book along 3 axes:

$$\boldsymbol{I} = \begin{bmatrix} I_{xx} & & \\ & I_{yy} & \\ & & I_{zz} \end{bmatrix}$$

Symmetric objects rotating around their center of mass have obvious axes, and a diagonal matrix for their inertia tensor. The "principal axes" are easy to see, they will give us that

$$\boldsymbol{L} = I_{axis} \boldsymbol{\omega}$$

In other words, the angular momentum is parallel to the angular velocity.

However, not all systems have obvious principal axes. For example, consider a cube of side length a and mass M rotating around its corner. If we find the inertia tensor for the cube:

$$\boldsymbol{I} = Ma^2 \begin{bmatrix} 2/3 & -1/4 & -1/4 \\ -1/4 & 2/3 & -1/4 \\ -1/4 & -1/4 & 2/3 \end{bmatrix}$$

So, if we rotate this cube around the $\hat{x}$ axis, $\boldsymbol{L}$ is not in the $\hat{x}$ direction:

$$\boldsymbol{L} = \boldsymbol{I}\boldsymbol{\omega} = Ma^2\omega \begin{bmatrix} 2/3 \\ -1/4 \\ -1/4 \end{bmatrix}$$

Thus we have that $\hat{x}$ is not a principle axis for the cube rotating around its corner. However, consider $\omega = \frac{\omega}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. In this case:

$$\boldsymbol{L} = \boldsymbol{I}\boldsymbol{\omega} = Ma^2 \begin{bmatrix} 1/6 \\ 1/6 \\ 1/6 \end{bmatrix} \frac{\omega}{\sqrt{3}}$$

and thus we have that this $\boldsymbol{\omega}$ is a principal axis, since the angular momentum is parallel to the axis of rotation.

Every inertia tensor has at least 3 principal axes, and has 3 orthogonal principal axes, which we can find using eigenvalue analysis. In other words, a different coordinate system can be found in which $\boldsymbol{I}$ is diagonal. We want to find $\boldsymbol{L}||\boldsymbol{\omega}$, so $\boldsymbol{I}\boldsymbol{\omega} = \lambda\boldsymbol{\omega}$, which we can rewrite as $(\boldsymbol{I} - \lambda\mathbf{1})\boldsymbol{\omega} = 0$.

For the cube, this becomes

$$\det \begin{bmatrix} 2/3-\lambda & -1/4 & -1/4 \\ -1/4 & 2/3-\lambda & -1/4 \\ -1/4 & -1/4 & 2/3-\lambda \end{bmatrix} = 0 \to \lambda^3 - 2\lambda^2 + \frac{55}{48}\lambda - \frac{121}{864} = 0 \to \left(\lambda - \frac{1}{6}\right)\left(\lambda - \frac{11}{12}\right)^2 = 0$$

Thus we have eigenvalues 1/6, 11/12, and 11/12. We have already found the eigenvector for 1/6, and if we plug in 11/12 for λ, and apply it against an unknown $\boldsymbol{\omega}$, we expect to get 0, and this finds the eigenvector. There are an infinite number of solutions for this case, so we can just take two orthogonal solutions.

4.1 Precession

When we have a bicycle wheel on a post, and the radial distance is D, and the wheel has mass m and rotation rate ω, the angular momentum around the post will be $I_{post}\omega$. In this case, $I = MR^2$, where we treat it like a torus, as all the mass is concentrated on the edge. Thus we have that $L = MR^2\omega$. The torque will be given by $DMg = \frac{dL}{dt}$.

From these, we can find the precession rate:

$$\Omega = \frac{d\phi}{dt} = \frac{DMg}{MR^2\omega} = \frac{Dg}{R^2\omega}$$

We see that larger ω means smaller Ω.

For a small gyroscope toy, where it is tipped, rather than held radially, the torque once again comes from the force of gravity, and the tip is given by an angle from vertical θ. We know that $\dot{\boldsymbol{L}} = \boldsymbol{\tau}$. In the body frame of the gyro rotor, $\boldsymbol{L} \approx \lambda_3 \omega \hat{e}_3$, where we are in some body frame coordinates $\hat{e}_1, \hat{e}_2$, and $\hat{e}_3$.

The effect of the torque will then be

$$\boldsymbol{\tau} = \lambda_3 \omega \dot{\hat{e}}_3$$

We can find that

$$|\dot{\hat{e}}_3| = \frac{|\boldsymbol{\tau}|}{\lambda_3 \omega} = \frac{mgD \sin\theta}{\lambda_3 \omega}$$

We then note that $|\dot{\hat{e}}_3| = \Omega \sin\theta$, where Ω is the rate at which ϕ goes around a circle, or the rate of precession.

4.2 Lamina

Suppose we want to calculate the inertia tensor for a lamina, or thin planar sheet. Suppose it is laying in the xy plane. We know that the z axis is guaranteed to be a principal axis, regardless of the shape in the xy plane. As a note, any axis parallel to a principal axis is also a principal axis.

Suppose we have a triangular lamina, with corners at $(1,0,0)$, $(0,1,0)$, and $(0,0,0)$.

We can compute terms in the inertia tensor:

$$I_{xy} = \int_{\text{shape}} -xy \, dm$$

Where in this case $dm = \sigma \, dx \, dy$, the mass of a small section of the lamina is given by the area and the mass per unit area. Thus the integral becomes

$$I_{xy} = -\sigma \int_0^1 dx \int_0^{1-x} xy \, dy$$

We would have to do the y integral first, due to the limit of integration, and then do the x integral.

4.3 General Rotational Dynamics

We know that torque changes angular momentum:

$$\frac{\partial \boldsymbol{L}}{\partial t} = \boldsymbol{\tau}$$

This relationship is true in the "space" frame, the inertial reference frame of the room or wherever we are. In the body frame, the reference frame fixed in the object with unit vectors $\hat{e}_1, \hat{e}_2$, and $\hat{e}_3$, which correspond to principal axes, we have a noninertial reference frame. We know the relationship between the space frame and the body frame, we can recall that

$$\left(\frac{d\boldsymbol{Q}}{dt}\right)_{iner} = \left(\frac{d\boldsymbol{Q}}{dt}\right)_{rot} + \boldsymbol{\Omega} \times \boldsymbol{Q}$$

For this kind of rotation, we generally use ω, rather than Ω, so we can write this out for the angular momentum:

$$\left(\frac{d\boldsymbol{L}}{dt}\right)_{iner} = \left(\frac{d\boldsymbol{L}}{dt}\right)_{rot} + \boldsymbol{\omega} \times \boldsymbol{L}$$

We know that the left term is the torque. Since we have chosen our basis vectors in the body frame to be along principal axes, the angular momentum in the body frame coordinates will be

$$\boldsymbol{L}_{rot} = \begin{bmatrix} \lambda_1\omega_1 \\ \lambda_2\omega_2 \\ \lambda_3\omega_3 \end{bmatrix}$$

Where the λs are the eigenvalues of the inertia tensor. Thus, we have that

$$\boldsymbol{\tau} = \begin{bmatrix} \lambda_1\dot{\omega}_1 \\ \lambda_2\dot{\omega}_2 \\ \lambda_3\dot{\omega}_3 \end{bmatrix} + \begin{bmatrix} \omega_2\omega_3(\lambda_3 - \lambda_2) \\ \omega_3\omega_1(\lambda_1 - \lambda_3) \\ \omega_1\omega_2(\lambda_2 - \lambda_1) \end{bmatrix}$$

We can now rewrite this as 3 equations:

$$\lambda_1\dot{\omega}_1 = (\lambda_2 - \lambda_3)\omega_2\omega_3 + \tau_1$$

$$\lambda_2\dot{\omega}_2 = (\lambda_3 - \lambda_1)\omega_3\omega_1 + \tau_2$$

$$\lambda_3\dot{\omega}_3 = (\lambda_1 - \lambda_2)\omega_1\omega_2 + \tau_3$$

These 3 equations are known as Euler's equations. They describe how $\boldsymbol{\omega}$ changes as seen in the body frame. Even if the torque is 0, $\boldsymbol{\omega}$ can change. This case is called "free precession".

4.3.1 Free Precession

In general, if $\lambda_1 \neq \lambda_2 \neq \lambda_3$, if the object is spinning with $\boldsymbol{\omega}$ close to a principal axis, lets say $\boldsymbol{\omega} \approx \omega_3\hat{e}_3$ with $\omega_1 \ll \omega_3$ and $\omega_2 \ll \omega_3$, then we can write out the Euler equations:

$$\dot{\omega}_3 = \frac{\lambda_1 - \lambda_2}{\lambda_3}\omega_1\omega_2$$

We see that $\omega_1\omega_2$ is two small numbers. On the other hand, if we look at

$$\dot{\omega}_1 = \frac{\lambda_2 - \lambda_3}{\lambda_1}\omega_2\omega_3$$

We see that this has only 1 small number. Thus we have that ω_3 is approximately constant, while ω_1 and ω_2 vary. We can take another derivative:

$$\ddot{\omega}_1 = \frac{\lambda_2 - \lambda_3}{\lambda_1}\dot{\omega}_2\omega_3$$

Where we used the fact that $\dot{\omega}_3 \approx 0$. Inserting the Euler equation for $\dot{\omega}_2$:

$$\ddot{\omega}_1 = \left(\frac{\lambda_2 - \lambda_3}{\lambda_1}\right)\left(\frac{\lambda_3 - \lambda_1}{\lambda_2}\right)\omega_3^2\omega_1$$

We see that this is similar to the harmonic oscillator if this constant term multiplied against ω_1 is negative. If it is negative, then ω_1 behaves as a harmonic oscillator with

$$\omega_{osc} = \omega_3\sqrt{-\left(\frac{\lambda_2 - \lambda_3}{\lambda_1}\right)\left(\frac{\lambda_3 - \lambda_1}{\lambda_2}\right)}$$

This tells us that $\boldsymbol{\omega}$ wobbles around $\hat{e}_3$ with frequency ω_{osc} in the body frame. However, this is only true if the factor is negative. If the factor is positive, then we have exponential growth, the rotation axis would be unstable.

Another case of this is if $\lambda_1 = \lambda_2 = \lambda_{12}$. In this case, Euler's equations become

$$\lambda_3 \dot{\omega}_3 = 0$$
$$\lambda_{12} \dot{\omega}_1 = (\lambda_{12} - \lambda_3)\omega_2 \omega_3$$
$$\lambda_{12} \dot{\omega}_2 = (\lambda_3 - \lambda_{12})\omega_3 \omega_1$$

If we once again assume that $\boldsymbol{\omega}$ is close to $\hat{e}_3$, we will find that

$$\ddot{\omega}_1 = \frac{-(\lambda_{12} - \lambda_3)^2}{\lambda_{12}^2} \omega_3^2 \omega_1$$

In this case, we have a stable oscillation, with

$$\omega_{osc} = \frac{|\lambda_{12} - \lambda_3|}{\lambda_{12}} \omega_3$$

The direction of the rotation/precession depends on whether $\lambda_{12} > \lambda_3$ or vice versa. We have that

$$\boldsymbol{\omega} = (\omega_0 \cos \Omega t, -\omega_0 \sin \Omega t, \omega_3)$$

And we can then compute the angular momentum:

$$\boldsymbol{L} = (\lambda_{12}\omega_0 \cos \Omega t, -\lambda_{12}\omega_0 \sin \Omega t, \lambda_3 \omega_0)$$

Let use note some key features. In the body frame, $\hat{e}_3$ is fixed, while $\boldsymbol{\omega}$ and $\boldsymbol{L}$ rotate (wobble) around it. In the space frame, there is no torque, so $\boldsymbol{L}$ is fixed, and $\boldsymbol{\omega}$ and $\hat{e}_3$ rotate (wobble).

In either frame, we note that L_3 is constant, along $\hat{e}_3$.

For a spinning gyroscope or top, the math is tedious, but we can work out the Lagrangian using the three Euler angles as coordinates, and use the standard tool, the Euler-Lagrange equation, to find the equations of motion for the coordinates. If we do out the tedious algebra, we have that the equation of motion for θ is given by

$$\lambda_{12}\ddot{\theta} = \lambda_{12}\dot{\phi}^2 \sin\theta \cos\theta - \lambda_3(\dot{\psi} + \dot{\phi}\cos\theta)\dot{\pi}\sin\theta + MgR\sin\theta$$

Where ψ is the rotation of the top along its axis, θ is the angle from the vertical, and ϕ is the sweep angle from some starting orientation. One special case of this is steady precession, motion with constant θ. Thus we have that $\dot{\theta} = \ddot{\theta} = 0$, and with constant $\dot{\phi}$ (which we relabel Ω):

$$0 = \lambda_{12}\Omega^2 \sin\theta\cos\theta - \lambda_3\omega_3\Omega\sin\theta + MgR\sin\theta$$

This is a quadratic equation for Ω, so we have two solutions for Ω, the two rotation rates which represent steady precession, without nutation. Nutation is when we allow θ to vary. We can analyze this using energy conseration, using the conserved quantities:

$$p_\psi = \lambda_3(\dot{\psi} + \dot{\phi}\cos\theta) = L_3$$
$$p_\phi = \lambda_{12}\sin^2\theta\dot{\phi} + L_3\cos\theta = L_z$$

We can use these to eliminate the $\dot{\phi}$ and $\dot{\psi}$ variables. We then have a differential equation for only $\theta(t)$, and we can describe this system using an effective potential, $U_{eff}(\theta)$:

$$\frac{1}{2}I\dot{\theta}^2 + U_{eff}(\theta) = E$$

Where E is a constant.

5 Chaos