Contents

1 Differential Equations

Differential equations relate changes to current values. We care because we want to track changes in values, and we want to make predictions about things in the physical world. Being able to say how things change is very important, and diffeqs are the main tool we use for that.

Differential equations have two types of variables. **Independent variables** are the input, and we have to take them for granted. **Dependent variables** represent the behavior of the system that we are interested in, and can be considered the output of the equation.

Differential equations require notation for writing derivatives. There are different types of notations. Leibniz notation is the fractional notation, ($\frac{dy}{dx}$). Newton's notation is the prime notation (y'). Operator notation ($D\,y$, $D_t\,y$, $D^2\,y$) uses D as an operator to differentiate what is to the right of it. We also have partial derivatives ($\frac{\partial y}{\partial x}$, $\partial_x y$). Newton also had special notations for time derivatives ($\dot{y}$, $\ddot{y}$).

We have seen many different examples of differential equations, the first and foremost being Newton's Second Law, $F = ma$, which is really just $F = m\ddot{x}$. From electromagnetism, we have Kirchoff's Voltage Law, $L\dot{I} + RI + \frac{q}{C} = \mathcal{E}$. We also have Newton's Cooling Law, $\frac{dQ}{dt} = kA\frac{dT}{dx}$. Maxwell's equations are 4 differential equations:

$$\nabla \cdot B = 0 \qquad \nabla \cdot E = \frac{\rho}{\mathcal{E}_0} \qquad \nabla \times E + \partial_t B = 0 \qquad \nabla \times B - \frac{1}{c}\partial_t E = \mu_0 J$$

Let's now go through these and flag the independent and dependent variables. In Newton's Second Law, we care about the position of some particle, and the position is evolving according to time. The independent variable is t, and the dependent variable is x. In the Voltage Law, time is once again independent, and the charge in the capacitor q is the dependendent variable (Note that I can be determined from q). In the heating and cooling law, x and t are independent variables, and the dependent variables are Q and T. In Maxwell's equations, we have 4 independent variables, $\vec{x}$ and t, and we have many dependent variables, $\vec{B}$, $\vec{E}$, $\vec{J}$, and ρ.

1.1 Terminology

A diffeq is **ordinary** if there is just 1 independent variable. The equation is a **partial** diffeq if it has more than 1 independent variable. For example, Newton's Second Law in 1 dimension is an ordinary diffeq, and Maxwell's Equations are partial diffeqs.

The **order** of a diffeq is the highest number of derivatives in the equation. For example, $F = m\ddot{x}$ is a second order diffeq.

A **linear** equation takes the form

$$ay + by' + cy'' + \cdots = f$$

And an equation is **nonlinear** if y is ever raised to a power that is not 1 (for example if it has a y^2 or y^5).

A **solution** to a differential equation is any function that when inserted as the dependent variable(s) yields an equality.

1.2 Methods of Solution

One of the first methods to solve a diffeq is to simply guess the answer, whether it be a random guess or through experience. Another method is to integrate, which we could use on these:

$$y' = c \qquad \ddot{y} = -g$$

Another method is to ask a computer. Computers can compute the answer algebraically (looking it up in a database), and numerically (giving a list of pairs x, $y(x)$).

1.3 First Order ODEs

A first order ODE is a diffeq that only includes the first derivative of the dependent variable. Here are some examples:

$$y' = \frac{x}{y} \qquad y' = y - x^2 \qquad y' = x - y^2$$

The first of these diffeqs is separable, the second equation is linear in y (no more than y^1), and the third equation is difficult (no algebraic solutions).

1.3.1 Separable Diffeqs

Separable 1st order ODEs can be solved via separation of variables:

$$y' = \frac{x}{y} \rightarrow \frac{dy}{dx} = \frac{x}{y}$$

From this point, we treat the differential as a fraction:

$$y\,dy = x\,dx$$

We can then integrate both sides:

$$\int y\,dy = \int x\,dx$$

We can now proceed in two ways. One way is to use the indefinite integrals:

$$\frac{y^2}{2} + A = \frac{x^2}{2} + B$$

$$y = \sqrt{x^2 + C}$$

The other method is to use definite integrals instead, by picking particular limits, noting that the integrand x and the limit x are different things

$$\int_{y(x_0)}^{y(x)} y\,dy = \int_{x_0}^{x} x\,dx$$

If we do these integrals out:

$$\frac{1}{2}(y^2 - y_0^2) = \frac{1}{2}(x^2 - x_0^2)$$

Doing this out, we can then solve for y. If we had specified the initial conditions, we could just go down the indefinite integral route and then insert the initial conditions.

1.3.2 Integral Curves

Say we have some diffeq $y' = f(x, y)$. If we plot on axes of x and y, the diffeq gives us the slope at every single (x, y). This then gives us the **direction field**. The solution is something called an **integral curve**, and is tangent to the line of the direction field at every point it goes through. If $y_1(x)$ is a solution to the differential equation, then its graph is an integral curve. Let's prove this.

Proof. The claim is that $y_1(x)' = f(x, y_1(x))$ for every x where y_1 is defined. The geometric claim is that the slope at $(x, y(x))$ is the slope of the direction field. The slope at $(x, y(x))$ is just y', and the slope of the direction field is just $f(x, y)$. Thus, we have that the solution graphs an integral curve. □

Generally, we draw direction fields via computers. Computers simply pick points in the range given, and compute $f(x, y)$ at each point, and then just draw it. This method is very tedious to do by hand, so we have a different method of doing it. We instead pick the slope we're interested in (call it c). We know that $y' = c = f(x, y)$. We can plot $f(x, y) = c$ for a few values of c. These curves are called **isoclines**. We can then draw in lots of lines with slope c along the isoclines (drawing in as many or as few as you want). Also note that integral curves cannot cross, which oftentimes forces the integral curves to take a certain shape. They can't cross due to the Existence and Uniqueness theorem:

Theorem 1.1. *$y' = f(y, x)$ has one and only one solution through (x_0, y_0). For there to be a solution, f must be continuous in the vicinity of (x_0, y_0) (geometrically, integral curves there exist). To get the uniqueness of the solution, $\partial_y f$ is continuous near (x_0, y_0) (geometrically, this implies that there is only one integral curve that goes through this point).*

1.3.3 Euler's Numerical Method

Suppose we have an IVP, with $y' = f(x, y)$ and $y(x_0) = y_0$. We first choose a step size h (Δx, dx). We can place the line element with the slope at the point (x_0, y_0), and then move forwards to $x_1 = x_0 + h$. We then project the slope we found until it intersects with the vertical line at x_1, which gives us the point (x_1, y_1). We can then evaluate $f(x_1, y_1)$ and repeat the process, placing down the slope, moving forwards by the step size, and project the line element to get the next point.

1. Start with (x_n, y_n)
2. Slope $A_n = f(x_n, y_n)$
3. Next x is $x_{n+1} = x_n + h$
4. Next y is $y_{n+1} = y_n + hA_n$

This method simply operationalizes the derivative, except gets rid of the limits and just works with a small step size to get a good enough approximation:

$$A_n = \frac{y_{n+1} - y_n}{h}$$

If our integral curve is a line, Euler's method will be exactly correct. If the solution is convex (concave up), Euler's method will underestimate. On the other hand, if the solution was concave down, Euler's method will overestimate the solution points. For the example $y' = x^2 - y^2$, we can implicitly differentiate, $y'' = 2x - 2yy'$. If we now plug in initial conditions such as $(0, 1)$, we can compute the second derivative, which is $y'' = 2$, telling us that Euler's method would underestimate the solution curve.

How can we get a better answer? The obvious answer is to take smaller steps, which will bring the results closer to the actual solution curve. Error is given by the absolute value of the difference between the exact answer and Euler's results. One can show that the error is proportional to the step size h, $E \sim h$. Another way to try and improve is to try to improve A_n. To do this, we can use higher order methods. These methods go through the process of using Euler's method, and then using pairs of slopes to provide an improvement (essentially looking at the second derivative in order to correct itself):

$$A_n = \left(\frac{A_n^3 + A_{n+1}^e}{2}\right)$$

This method has multiple names, Heun's Method, Improved Euler's Method, Modified Euler's Method, and Runge-Kutta 2. This method has the error being quadratic in regards to the stepsize, $E \sim h^2$. Runge-Kutta has other variants, such as RK4, which has $E \sim h^4$.

Numerical methods do have their drawbacks. One of them is numerical instability, which is where we run into issues with the precise storage of numbers (think about storing things as floating point numbers and how that isn't always accurate). Another issue is singular points. If we take the equation $y' = y^2$, which has the solution $y = \frac{1}{c-x}$. This looks innocous, but if we graph a solution, we have that $x = c$ is a singular point, as the solution goes from infinity to negative infinity on either side. If we use any numerical method, after the point, the solution the numerical method will produce will be completely wrong, no matter the step size we use. Since we can't see the singular points from looking at the differential equation, we can't really do much.

1.3.4 First-Order Linear ODEs

A generic first-order linear ODE takes the form

$$a(x)y'(x) + b(x)y(x) = c(x)$$

where x is the independent variable, and a,b, and c are arbitrary functions. We see that the equation is linear in y and y', i.e. there are no y^2, e^y, etc. This equation is **homogeneous** if $c(x) = 0$. The **Standard Linear Form** is written as

$$y' + p(x)y = q(x)$$

There are many physical models that follow this, some of which include temperature, concentration, radioactive decay, mixing, and bank interest.

Let's look at the conduction-diffusion model. If we have a bath of some liquid that is being heated, with some temperature T_l. If we place a sealed container in the bath, we want to find the temperature inside of the the sealed container. The temperature changes according to Newton's Law of Cooling:

$$\frac{dT}{dt} = k(T_l - T)$$

Another situation for this model is if we're talking about salt concentrations in a box with a semipermeable membrane:

$$\frac{dC}{dt} = k(C_l - C)$$

We can see that we can rewrite these in the standard linear form:

$$\frac{dC}{dt} + kC = kC_l$$

If we look at the standard linear form, we can solve this by guessing/using experience. We multiply both sides by some function U:

$$Uy' + Upy = Uq$$

The aim is to get something that looks like $\frac{d}{dx}(\dots) = Uq$. If we had $\frac{d}{dx}(Uy) = Uq$, we'd expand the left side and get $U'y + Uy'$ by the product rule. Matching this to our equation, we see that we need a function such that

$$\frac{dU}{dx} = pU \to \frac{dU}{U} = p\,dx$$

This then gives us that

$$\boxed{U = e^{\int p\,dx}}$$

This is known as the method of the integrating factor. Let's do an example.

If we have the equation $xy' - y = x^3$, we can put it in standard linear form:

$$y' - \frac{1}{x}y = x^2$$

This gives us that $p(x) = \frac{-1}{x}$. Integrating this:

$$\int p\,dx = -\ln|x| + C$$

Thus we know that

$$U = e^{\int p\,dx} = e^{-\ln x + C} = \frac{e^c}{x}$$

Multiplying both sides by U, we see that we can just drop the e^c constant, because we cna divide it out from both sides:

$$\frac{1}{x}y' - \frac{1}{x^2}y = x$$

We can now undo the product rule on the left side (its always Uy):

$$\frac{d}{dx}\left(\frac{y}{x}\right) = x$$

Now integrating:

$$\int d\left(\frac{y}{x}\right) = \int x\,dx$$

Which gives us that

$$\frac{y}{x} = \frac{x^2}{2} + C$$

Giving us the final answer that

$$y = \frac{x^3}{2} + Cx$$

Let's do another example. If we have the diffeq

$$(1 + \cos x)y' - y\sin x = 2x$$

We can put this in standard linear form:

$$y' - \frac{\sin x}{1+\cos x}y = \frac{2x}{1+\cos x}$$

We see that $p(x) = -\frac{\sin x}{1+\cos x}$, so we have to compute the integral:

$$\int \frac{-\sin x}{1+\cos x}\,dx = \ln(1+\cos x)$$

This gives us that $U = 1+\cos x$. Multiplying both sides by U:

$$(1+\cos x)y' - \sin x y = 2x$$

We ended up back where we started! But now we know that the left side can be written as a product rule:

$$\frac{d}{dx}\left[(1+\cos x)y\right] = 2x$$

This then gives us that

$$y = \frac{x^2+C}{1+\cos x}$$

Now let's look at this method for the conduction-diffusion model, restricted to the case where the coefficients are constant (in this case k). We know that $U = e^{\int k\,dt} = e^{kt}$, and we multiply both sides by the integrating factor:

$$e^{kt}\dot{T} + ke^{kt}T = kT_l e^{kt}$$

Rewriting,

$$\frac{d}{dt}\left(e^{kt}T\right) = kT_l e^{kt}$$

Rewriting and integrating:

$$T = e^{-kt}\left[\int kT_l e^{kt}\,dt + C\right]$$

We will then put limits on this integral to make it a little more descriptive of our situation. If we have some initial conditions $t = 0 \to T(0) = T_0$:

$$T = e^{-kt}\left[\int_0^t kT_l e^{kt}\,dt + C\right]$$

We can see that the initial conditions control the **transient** part of the solution, the part that goes away after a long period of time (in this case its because of the outside exponential decay). The **steady-state** solution is controlled by T_l.

What is so special about these linear ODEs? If we take a generic linear first order ODE:

$$y' + p(x)y = q(x)$$

We can generalize this with more terms:

$$\dots y''' \dots y'' \dots + y' + p(x)y = q(x)$$

These diffeqs obey the Superposition Principle:

Theorem 1.2. *If q_1 produces the solution y_1 and q_2 produces the solution y_2, then $q_1 + q_2$ produces the solution $y_1 + y_2$, and cq_1 produces cy_1.*

This principle does not work if we have any terms that are nonlinear, such as y^2.

Let's try and find an application for this. If we have a system with some trig input (which often arises in physics):

$$y' + ky = kq(t) \qquad q(t) = \cos \omega t$$

where $\omega = \frac{2\pi}{T}$, where T is the period, and ω is the angular frequency.

To solve this, we "complexify", a trick that we will use a lot. Recall that $e^{i\omega t} = \cos \omega t + i \sin \omega t$. We will write a different diffeq, one for a complex y:

$$z' + kz = ke^{i\omega t}$$

After we solve this equation, we will just take $y = Re(z)$, and we will be able to solve the original equation that we are interested in. We can solve the complex equation using the integrating factor, with $U = e^{kt}$:

$$e^{kt}z' + ke^{kt}z = ke^{kt}e^{i\omega t}$$
$$\frac{d}{dt}\left[e^{kt}z\right] = ke^{kt}e^{i\omega t}$$

Integrating and solving for z:

$$ze^{kt} = \frac{ke^{(k+i\omega)t}}{k + i\omega} + C$$

Rewriting, we will find that

$$z = \frac{1}{1 + \frac{i\omega}{k}}e^{i\omega t} + Ce^{-kt}$$

This is the solution for the complex diffeq, so we need to take the real portion. One nice way to do this is to convert to polar. If we let $\alpha = 1 + \frac{i\omega}{k}$:

$$\alpha = \sqrt{1 + \left(\frac{\omega}{k}\right)^2}e^{i\phi}$$

and thus

$$z = \frac{1}{\sqrt{1 + \left(\frac{\omega}{k}\right)^2}}e^{i(\omega t - \phi)} + Ce^{-kt}$$

Taking the real part of this:

$$y = \frac{1}{\sqrt{1 + \frac{\omega^2}{k^2}}}\cos(\omega t - \phi) + Ce^{-kt}$$

where ϕ is the phase shift/lag ($\tan \phi = \frac{\omega}{k}$). If we now plot this solution, we see that we have a sinusoid, shifted to the right, as ω and k are both positive. Therefore, ϕ is positive, shifting the solution to the right. ϕ modulates the shift between the peaks of the input function and the solution.

If we increase k, physically it means that the heat conduction increases, and so we would expect the solution to track the input function more closely, which checks out, as $\frac{\omega}{k}$ decreases, decreasing $\tan \phi$, sending the shift towards 0. We also see that the amplitude increases. Let's now consider the other case, where k is very low. We expect the inside of the container to barely change even as we heat it up. We see that $\frac{\omega}{k}$ increases, meaning that the amplitude of the solution function decreases, and $\tan \phi$ increases, with ϕ maxing out at $\frac{\pi}{2}$. Thus the shift will never be worse than a quarter of the period.

1.3.5 Scaling

One trick that we can use is scaling:

$$\tilde{x} = \frac{x}{a} \qquad \tilde{y} = \frac{y}{b}$$

This is often useful when changing units, making variables dimensionless, or reducing/simplfying the constants in a problem. As an example, where M and T are both temperatures:

$$\frac{dT}{dt} = k\left(M^4 - T^4\right)$$

This is similar to the Stefan-Boltzmann law. We can scale the variables to make T dimensionless:

$$\tilde{T} = \frac{T}{M} \to T = M\tilde{T}$$

Inserting this into the equation:

$$\frac{d}{dt}(M\tilde{T}) = k(M^4 - M^4\tilde{T}^4)$$

Simplifying:

$$\frac{d}{dt}\tilde{T} = kM^3(1 - \tilde{T}^4)$$

We can then clump kM^3 into a single constant, $\tilde{k}$. We could then solve this equation and then restore the constants/scaling. There are two types of substitutions like this. **Direct Substitutions** are when we make new variables that are functions of old variables ($\tilde{T} = \frac{T}{M}$), and **Inverse Substitutions** are when old variables are functions of new variables ($T = M\tilde{T}$). We have run into these when we compute integrals, like when we do u-substitutions, which are direct substitutions.

Let's do an example of a direct substitution, the Bernoulli Equation, which is nonlinear for $n > 1$:

$$y' = p(x)y + q(x)y^n$$

Note that there are no pure $f(x)$ or constant terms in the equation. The trick to solve this equation is to divide by y^n:

$$\frac{y'}{y^n} = \frac{p(x)}{y^{n-1}} + q(x)$$

We will then make a substitution, $v = \frac{1}{y^{n-1}}$: We can take the derivative:

$$v' = (1-n)\frac{1}{y^n}y'$$

Plugging in our substitutions:

$$\frac{1}{1-n}v' = p(x)v + q(x)$$

We can solve this equation, as it is linear in v. We can then undo the substitution:

$$y = \sqrt[n-1]{\frac{1}{v}}$$

1.3.6 Homogenous ODEs

If we have a diffeq of the form

$$y' = f(y,x) = F(\frac{y}{x})$$

For example:

$$y' = \frac{x^2 y}{x^3 + y^3}$$

If we divide by x^3:

$$y' = \frac{\frac{y}{x}}{1 + \left(\frac{y}{x}\right)^3}$$

Another example is $xy' = \sqrt{x^2 + y^2}$, which we can rewrite:

$$y' = \sqrt{1 + \left(\frac{y}{x}\right)^2}$$

These are invariant under "zooming in", or rather they are scalable as long as both x and y share the same scaling factor:

$$x = a\tilde{x} \qquad y = a\tilde{y}$$

Which gives us that

$$dx = a\,d\tilde{x} \qquad dy = a\,d\tilde{y}$$

We substitute in $z = \frac{y}{x} = \frac{\tilde{y}}{\tilde{x}}$:

$$\frac{dy}{dx} = F(\frac{y}{x}) = F(\frac{a\tilde{y}}{a\tilde{x}}) = F(z)$$

The left side of the equation becomes:

$$z'x + z = F(z)$$

This is now solvable via separation of variables:

$$\frac{dz}{F(z) - z} = \frac{dx}{x}$$

Any time we see a homogenous equation like this, we can use the substitution $z = \frac{y}{x}$, and convert it into a linear diffeq. Generally, we only really know how to solve linear diffeqs, we just find tricks to convert nonlinear diffeqs into linear diffeqs, which we can solve.

1.3.7 Autonomous ODEs

Autonomous first-order ODEs have no explicit independent variable:

$$y' = f(y)$$

Suppose f is a very unpleasant function. Can we still get qualitative information about the solution?

If we think about the isoclines of this diffeq, they all have some slope $c = f(y)$, we have isoclines that are horizontal lines. If we draw some integral curves, we see that they are invariant under translation, they're all the same curves just shifted to the left or right. y_0 is a **critical point** if

$f(y_0) = 0$ (basically if the direction field is flat there). If the isocline is horizontal, and the slope of the direction field is horizontal, then the horizontal line is a solution/integral curve:

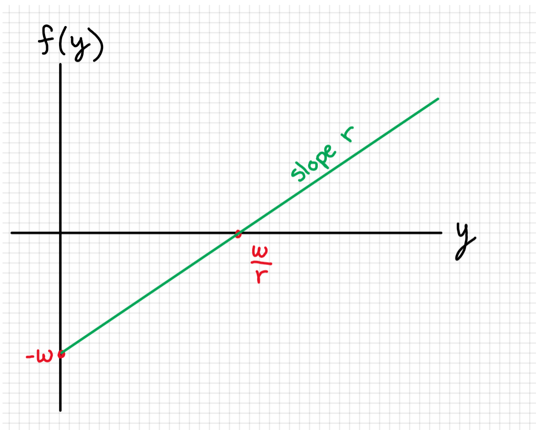
$$\frac{d}{dx}(y_0) = 0 \qquad f(y_0) = 0$$

Suppose we know some critical points, y_0 and y_1. This gives us two integral curves, and in between, the curves have to go between them, and they cannot cross. This gives us a method of sketching the solution to autonomous diffeqs.

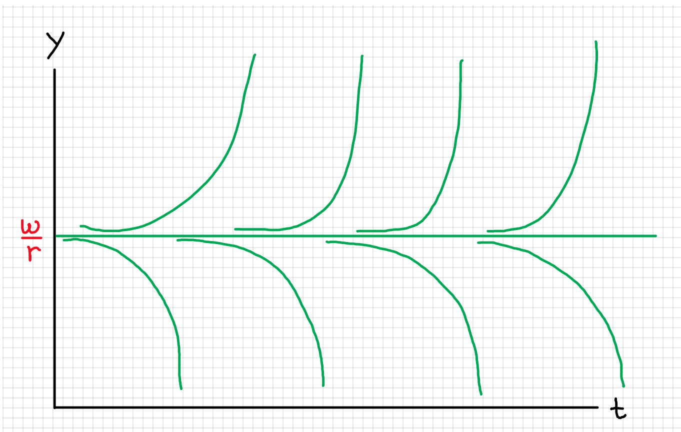
Let's do an example. Let's say we have some deposit into a bank account. Let y be the money in the account, and r is the interest rate. With no embezzlement, we have the diffeq $\dot{y} = ry$, which gives us exponential growth. However, say we have some embezzlement going on, and everyday they steal some constant amount of money:

$$\dot{y} = ry - w$$

If we sketch $f(y)$ as a function of y, we have something that looks like

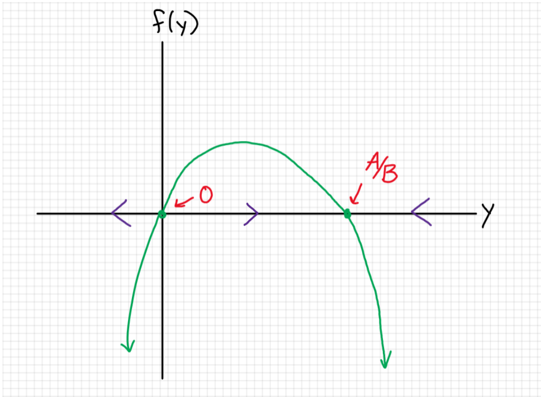


If $y > \frac{w}{r}$, y has an upward flow, and if $y < \frac{w}{r}$, we have a flow towards less and less y. The point $\frac{w}{r}$ is the critical point. Let's now draw y as a function of x:

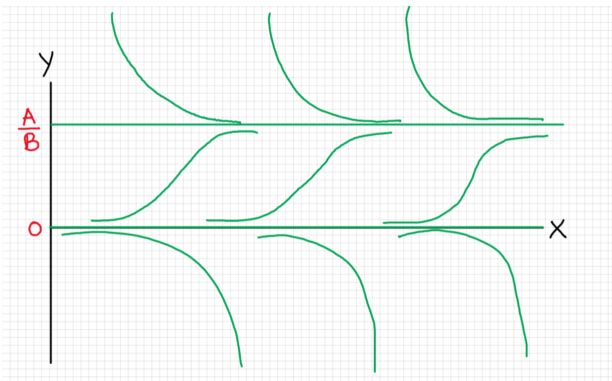


We can see that the bank embezzler could steal at a rate of $\frac{w}{r}$ and keep the bank balance steady.

Let's look at logistic growth. Classical exponential growth follows the differential equation $\frac{dy}{dt} = ky$, but logistic growth takes the form $\frac{dy}{dt} = ay - by^2$. This is the same as saying $k = a - by$. We can once again plot $f(y)$ against y:



We can now put the critical points into the second plot:



Let's now do logistic growth with harvesting. This follows the diffeq $\frac{dy}{dt} = ay - by^2 - h$. When $h = 0$, we have just regular logistic growth as we've seen. In the case where $h > 0$ but small: In the case where $h \gg 0$: In the case where h is perfect, and we have a double root:

1.4 Second Order ODEs

Most dynamical systems in physics follow some form of second order ODEs, and thus solving them is very important. The main motivation for studying these ODEs is the equation

$$F = ma$$

This is a second order ODE, because the acceleration is the second derivative of the position of the object:

$$F = m\ddot{x}$$

Here is a list of equations that will show up a lot:

1. Wave equation
2. Harmonic oscillator
3. Diffusion
4. Electromagnetism

The standard form of writing a linear second order ODE is

$$y'' + p(x)y' + q(x)y = f(x)$$

We see that once again y and its variants is only raised to the first power. We can classify these ODEs into homogeneous and inhomogeneous equations, where homogeneous equations are of the form $f(x) = 0$. Inhomogeneous equations are of the form $f(x) = 0$. $f(x)$ is often called the force, the input, the signal, or the driver/driving term. This is the part of the equation that the experimenter can control. Let's look at the simplest case of second order ODEs.

1.4.1 Homogeneous, Constant Coefficients

Say we have a spring/mass/dashpot system. The spring has spring constant k, and the dashpot has constant c. Dashpots are like the pistons that stop doors from slamming. These three parts give us the differential equation:

$$m\ddot{x} = -kx - cv$$

We can rearrange this, and we find that

$$\ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = 0$$

We see that this takes the form

$$y'' + Ay' + By = 0$$

where A and B are both constants and the equation is homogeneous. To solve this, we assume that the general solution takes the form

$$y = c_1 y_1 + c_2 y_2$$

where we can adjust c_1 and c_2 to match any initial conditions. We have two initial conditions, $y(t_0)$ and $y'(t_0)$. The guess that we're going to make is an exponential (just based on experience and physical intuition). We can guess

$$y = Ce^{rt}$$

Inserting this solution:

$$r^2 e^{rt} + Are^{rt} + Be^{rt} = 0$$

This looks bad, but we can just divide out the e^{rt} because it is never 0:

$$r^2 + Ar + B = 0$$

We can then solve this quadratic (the characteristic equation) to get the value of r in terms of A and B. If our value for r solves the characteristic equation, then our guess $y = e^{rt}$ solves the ODE. We have two solutions:

$$r_{\pm} = \frac{-A \pm \sqrt{A^2 - 4B}}{2}$$

This then gives us our two solutions:

$$y = C_+ e^{r_+ t} + C_- e^{r_- t}$$

Looking at the value of r, we have 3 cases. One is where we have real and distinct roots, another is when we have complex roots, and the last situation is when we have real and equal roots.

Let's look at the first case, where we have two distinct real roots. If we have the equation $y'' + ry' + 3y = 0$, we can see that we have the characteristic equation $r^2 + 4r + 3 = 0$, which we can factor $((r+3)(r+1) = 0)$, giving us the general solution

$$y = C_1 e^{-t} + C_2 e^{-3t}$$

Suppose we have the initial conditions $y(0) = 1$ and $y'(0) = 0$. If we now compute $y(0)$:

$$y(0) = C_1 + C_2 = 1$$

And we can compute $y'(0)$:

$$y'(0) = -C_1 - 3C_2 = 0$$

We now have a system of equations that we can solve. We find that $C_1 = -\frac{1}{2}$ and $C_2 = \frac{3}{2}$. Now we have the specific solution for this case. Plotting this, we see that we don't have any oscillation, as the dashpot is strong enough to overpower the spring.

Now let's look at the second case, where we have 2 complex roots, $r = a \pm ib$, (this implcitly tells us the characteristic polynomial). We can guess our solution:

$$y = c_+ e^{(a+ib)t} + c_- e^{(a-ib)t}$$

We have a theorem that helps us:

Theorem 1.3. *Consider the differential equation $y'' + Ay' + B = 0$. If $A, B \in \mathbb{R}$ and $y = u + iv$ is a solution to the differential equation, then u and v are both solutions to the differential equation.*

Proof. We can prove this by just plugging it in:

$$(u + iv)'' + A(u + iv)' + B(u + iv) = 0$$

If A and B are real, we can take the real part of the equation:

$$u'' + Au' + Bu = 0$$

And take the imaginary part:

$$v'' + Av' + Bv = 0$$

□

We can rewrite our guess (using the identity given via the triangle on homework 1):

$$y = e^{at}\left(c_+ e^{bit} + c_- e^{-bit}\right) = e^{at}\alpha\cos(bt - \phi)$$

We have 4 unknowns and only 2 initial conditions, so how do we get the values of the constants? One way is to just write everything out, but this isn't smart. There is a trick that we can use, and that is to require that $y = y^*$, which gets us that the y is real. This sends i to $-i$:

$$y^* = e^{at}\left(c_+^* e^{-ibt} + c_-^* e^{ibt}\right)$$

If we now say that $y = y^*$, we have to match terms, so we see that $c_+ = c_-^*$ and $c_- = c_+^*$. We can rewrite $c_+ = ce^{i\phi}$, and similarly for c_-:

$$y = ce^{at}\left(e^{i(bt+\phi)} + e^{-i(bt-\phi)}\right) = 2ce^{at}\cos(bt - \phi)$$

Let's do an example. If we have the equation $y'' + 4y' + 5y = 0$, we can write down the characteristic equation:"

$$r^2 + 4r + 5 = 0$$

Solving this, we see that $r = -2 \pm i$. We know the solution then:

$$y = e^{-2t}\left[c_1 \cos t + c_2 \sin t\right]$$

Suppose we had specified some initial conditions, such as $y(0) = 1$ and $y'(0) = 0$. WE can then use the solution and these conditions to fix c_1 and c_2:

$$e^0\left[c_1 \cos 0 + c_2 \sin 0\right] = y(0) = 1$$

This condition gets us that $c_1 = 1$. To use the second condition, we just need to differentiate:

$$y'(0) = 0 = -2e^0\left[\cos 0 + c_2 \sin 0\right] + e^0\left[-\sin 0 + c_2 \cos 0\right]$$

This leaves us with

$$0 = -2 + c_2$$

Which tells us that $c_2 = 2$. We could now use the triangle, with side lengths 1, 2, and $\sqrt{5}$:

$$y = \sqrt{5}e^{-2t}\cos(t - \phi)$$

Now let's look at the case where we have two equal, real roots. We have $r = -a$, where $a > 0$ because of physical intuition. This gets us the characteristic equation

$$r^2 + 2ar + a^2 = 0$$

We could write the ODE:

$$y'' + 2ay' + a^2y = 0$$

We can guess the solution:

$$y = ce^{-at} + de^{-at} = (c + d)e^{-at} = \tilde{c}e^{-at}$$

We see that we have just one solution? If we have an IVP, with $y(0) = 1$ and $y'(0) = 0$, we can plug in the first condition and find that $\tilde{c} = 1$. We can then use the second condition, and we find that $a\tilde{c} = 0$. This is an issue, because we end up with a contradiction if a isn't 0! Something about our assumption is wrong, and it turns out that it's our guess.

Let's do this so that we get a correct solution with two arbitrary constants, a method called reduction of order. We have the equation $y'' + 2ay + a^2y = 0$. We know that e^{-at} is a solution. Given a solution $y = e^{-at}$, we can guess another solution ue^{-at} where u is some function of t. We now want to find what u needs to be in order to solve the diffeq. We can take our new solution and differentiate it repeatedly:

$$y = e^{-at}u \qquad y' = -ae^{-at}u + u'e^{-at} \qquad y'' = a^2e^{-at}u - 2ae^{-at}u' + e^{-at}u''$$

We can now take these to plug into the differential equation:

$$a^2e^{-at}u - 2a^2e^{-at}u + 2ae^{-at}u' + a^2e^{-at}u - 2ae^{-at}u' + e^{-at}u'' = 0$$

Simplifying:

$$0 = 0u + 0u' + e^{-at}u''$$

We have a diffeq for u now. If ue^{-at} is a solution to our diffeq, then $u'' = 0$. The solution to this equation is just a line:

$$u = c_1 t + c_2$$

We can now take u and insert it back in:

$$y = (c_1 t + c_2)e^{-at}$$

Let's look at the undamped spring-mass-dashpot oscillation equation. We have that

$$\ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = 0$$

Generally we call $\frac{k}{m} = \omega_0^2$, and call $\frac{c}{m} = 2b$. The factor of 2 is there so that it condenses the equation into:

$$y'' + 2by + \omega_0^2 y = 0$$

Which gives us the characteristic equation

$$r^2 + 2br + \omega_0^2 = 0$$

Solving this with the quadratic equation:

$$r = \frac{-2b \pm \sqrt{16b^2 - 4\omega_0^2}}{2} = -b \pm \sqrt{b^2 - \omega_0^2}$$

We see that the factor of 2 makes the result for r look cleaner. In the undamped case, $b = 0$, which gives us that $y'' + \omega_0^2 y = 0$, and $r = \pm\sqrt{-\omega_0^2} = \pm i\omega_0$. This tells us that the solutions are

$$y = c_1 \cos\omega_0 t + c_2 \sin\omega_0 t = A\cos(\omega_0 t - \phi)$$

where $\tan\phi = \frac{c_2}{c_1}$.

1.5 Phase Space

If we have the equation we just solved:

$$\ddot{x} + \omega_0^2 x = 0 \qquad x = A\cos(\omega_0 t - \phi)$$

Differentiating our solution:

$$\dot{x} = -A\omega_0 \sin(\omega_0 t - \phi)$$

Phase space is a way to draw solutions, by plotting with the solution on the x axis and the derivative on the $\dot{x}$ axis. In this case, we will plot ellipses because of the ω_0 term in the derivative, so we instead plot against the $\frac{\dot{x}}{\omega_0}$ axis in the vertical direction, which in this case gives us concentric circles around the origin. We can use these to get meanings for A and ϕ. We can take $\omega_0 t - \phi$:

$$\omega_0 t - \phi = \omega_0(t - \frac{\phi}{\omega_0}) = \omega_0(t - t_0)$$

A gives us the radius of the circle, and $\frac{\phi}{\omega_0}$ gives us the starting time. Suppose we had the initial condition $x(t_0) = A$ and $\dot{x}(t_0) = 0$. This point will be located at $(A, 0)$. Immediately after this t_0, x will decrease and $\dot{x}$ will also decrease, which tells us that we go around the circles clockwise. We can notice that any point in the phase space is on exactly one curve. These curves are often called phase-space trajectories.

If we look at the damped spring-mass-dashpot system, we have that

$$r = -b \pm \sqrt{b^2 - \omega_0^2}$$

When do we get oscillations? We get oscillations when r is complex, which is when $b^2 - \omega_0^2 < 0$. For physical problems, $b > 0$ and $\omega_0 > 0$. Thus we need $b < \omega_0$. We can rewrite r:

$$\begin{aligned} r &= -b \pm \sqrt{-(\omega_0^2 - b^2)} \\ &= -b \pm i\omega \end{aligned}$$

where $\omega^2 = \omega_0^2 - b^2$. ω is sometimes called the pseudofrequency (in mathematics). The general solution for this case is

$$Ae^{-bt}\cos(\omega t - \phi)$$

This is a decaying oscillation. Note that the zeroes of the function would be evenly spaced.

Plotting the damped equation in phase space, we want to plot x and $\dot{x}$. In this case, we scale the $\dot{x}$ axis into $\frac{\dot{x}}{\omega}$. If we plot the damped solution, we are left with an inward spiral, moving clockwise. If b is very small, the spiral takes a while to reach the center, while if b is large, then the solution very quickly approaches the center. If we set $b = 0$, then we are just left with the circles we saw previously.

Let's look at the intuition for the spring-mass-dashpot system. We have the equation

$$\ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = 0$$

We rewrite this

$$\ddot{x} + 2b\dot{x} + \omega_0^2 x = 0$$

We found that the solution is of the form

$$x = Ae^{-bt}\cos(\omega t - \phi)$$

We see that the solution has 4 unknowns, two of which come from the diffeq itself. b controls the damping, and ω is based on ω_0 and b. These two are based on physical properties of the system, and are not based on initial conditions. On the other hand, A and ϕ depend entirely on the initial conditions, and do not come from the ODE at all.

1.6 General Second-Order Linear Homogeneous ODES

We will now lift the restriction that we have constant coefficients, but we retain linearity. These equations take the form

$$y'' + p(x)y' + q(x)y = 0$$

We can look for two solution y_1 and y_2 that are independent of each other ($y_1 \neq cy_2$). If we have these two solutions, then any linear combination $c_1y_1 + c_2y_2$ is also a solution. This can be shown via linearity. Another interesting result is that any solution can be written as this linear combination.

Let's show that if y_1 and y_2 are solutions, then any linear combination of them is a solution. We can do this by just plugging what we claim is also a solution into the differential equation.

$$(c_1y_1 + c_2y_2)'' + p(x)(c_1y_1 + c_2y_2)' + q(x)(c_1y_1 + c_2y_2) = 0$$

We can use the properties of derivatives:

$$c_1y_1'' + c_2y_2'' + p(x)(c_1y_1' + c_2y_2') + q(x)(c_1y_1 + c_2y_2) = 0$$

Rearranging terms:

$$c_1(y_1'' + p(x)y_1' + q(x)y_1) + c_2(y_2'' + p(x)y_2' + q(x)y_2) = 0$$

The claim is that y_1 solves the diffeq, and thus the first term is 0. We also claim that y_2 solves the diffeq, and thus the second term is also 0. We are left with $0 = 0$, and thus the superposition is a solution.

Let's talk about linear operators. We have seen that $y'' + p(x)y' + q(x)y' = 0$ can be rewritten $D^2y + p(x)Dy + q(x)y = 0$. The beauty of this notation is that we can rewrite this in shorthand:

$$(D^2 + pD + q)y = 0$$

We now have an equation of the form

$$Ly = 0$$

Where L is an operator, something that takes in a function and outputs another function. The question is what y can we pick so that 0 comes out of the operator. L is a linear operator because it obeys the following properties:

1. $L(u + v) = Lu + Lv$
2. $L(cu) = cLu$

An example of a linear operator is differentiation.

Theorem 1.4. $\{c_1y_1 + c_2y_2\}$ *is enough to satisfy any initial conditions.*

Proof. Suppose we have the initial conditions $y(x_0) = a$ and $y'(x_0) = b$. We claim that $c_1y_1(x_0) + c_2y_2(x_0) = a$ and $c_1y_1'(x_0) + c_2y_2'(x_0) = b$. If we have the equations

$$c_1y_1(x_0) + c_2y_2(x_0) = a$$

$$c_1y_1'(x_0) + c_2y_2'(x_0) = b$$

We can write these in a matrix form, and the Wronskian at x_0 is determined by the determinant of the matrix:

$$W(y_1, y_2)(x_0) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}_{x_0}$$

Suppose that $y_1 = cy_2$. In this case, the determinant will be 0, and thus we can't solve the system of 2 unknowns. $\square$

Theorem 1.5. *If y_1 and y_2 are solutions to the ODE $Ly = 0$, either $W(y_1, y_2) = 0$ for all x, or $W(y_1, y_2)$ is never 0 for any x.*

1.7 Normalized Solutions

To normalize solutions, we pick a time, for example, $t = 0$. We have two solutions, Y_1 and Y_2, and they must satisfy special initial conditions:

$$Y_1(0) = 1 \quad Y_2(0) = 0 \qquad Y_1'(0) = 0 \quad Y_2'(0) = 1$$

If we plot this in phase space, we see that they plot out unit vectors.

Let's do an example, with $y'' + y = 0$. The solutions are

$$y_1 = \cos x \qquad y_2 = \sin x$$

We see that these are already normalized solutions. If we have $y'' - y = 0$, we have the solutions $y_1 = e^x$ and $y_2 = e^{-x}$. We want $Y_1(0) = 1 = c_1 + c_2$. We also want $Y_1'(0) = 0 = c_1 - c_2$. We will see that

$$Y_1 = \frac{e^x + e^{-x}}{2} \qquad Y_2 = \frac{e^x - e^{-x}}{2}$$

We see that we have sinh and cosh. This makes solving the IVP really easy.

Let's now talk about the Existence and Uniqueness Theorem. We had one for first order solutions, where we guaranteed the existence of a unique solution given inital conditions. For second order equations:

Theorem 1.6. *If p and q are continuous for all x, then there is one and only one solution for a given set of initial values (y_0, y_0') for the equation*

$$y'' + py' + qy = 0$$

The solution to this equation is $y_0 Y_1 + y_0' Y_2$.

1.8 Inhomogeneous Second Order Linear ODEs

The equations take the form

$$y'' + p(x)y' + q(x)y = f(x)$$

The homogeneous version of this equation is $y'' + p(x)y' + q(x)y = 0$. f is known by many names, such as the force, the input, the signal, or the external force. y is often called the motion, the output, or the response. The case where $f(x) = 0$ is known as the associated homogeneous equation, or the reduced equation. The soltuion to the associated equation is often known as the complementary solution (y_c) or the solution to the homogeneous equation (y_h). Let's look at some examples, such as the driven spring-mass-dashpot system or the driven RLC circuit. If we look at the RLC circuit, Kirchoff's voltage law tells us that $\sum V = 0$:

$$L\dot{I} + IR + \frac{q}{C} = \mathcal{E}$$

How do we go about solving these equations?

Theorem 1.7. *Suppose we have the differential equation*

$$Ly = f$$

The solution can always be written in the following way:

$$y = y_p + y_c$$

In other worse:

$$y = y_p + c_1y_1 + c_2y_2$$

Remember that $y_c = c_1y_1 + c_2y_2$ solve the associated homogeneous equation. y_p is known as the particular solution, and it refers to being particular to f.

The first step is to solve the homogeneous equation. We then want to find any particular solution.

Proof. Inserting the solution into the equation:

$$L(y_p + c_1y_1 + c_2y_2) = L(y_p) + c_1L(y_1) + c_2L(y_2)$$

Suppose we had 2 particular solutions, u and v (which means that $L(u) = f$ and $L(v) = f$). This means that $L(u - v) = L(u) - L(v) = f - f = 0$. □

We can make an analogy to a first order ODE with constant coefficients:

$$y' + ky = q(t)$$

We can do this with the integrating factor, and we find that

$$y = \left(e^{-kt} \int q(t)e^{kt}\,dt\right) + Ce^{-kt}$$

The term Ce^{-kt} is analogous to the complementary solution, and the first term is the particular solution. If we have that $k > 0$, we have that the complementary solution is transient, and the particular solution is the steady-state solution.

Let's talk about the transient solutions in the second order ODE with constant coefficients:

$$y'' + Ay' + By = f(t)$$

When can we separate into transient and steady-state solutions? We are guaranteed that the solution will be of the form

$$y = y_p + c_1y_1 + c_2y_2$$

with c_1 and c_2 being fixed by the initial conditions. We know that we have 3 cases of roots for the complementary solution:

$$c_1e^{r_1t} + c_2e^{r_2t}$$

$$(c_1 + c_2t)e^{rt}$$

$$e^{at}(c_1 \cos bt + c_2 \sin bt)$$

When do these give us stability (go to 0 at long times)? The first case goes to 0 if $r_1 < 0$ and $r_2 < 0$. In the critically damped case, we need $r < 0$, and in the complex conjugate case we need $a < 0$. Notice that all the roots have negative real parts.

Let's do some examples. If we take the equation

$$y'' + Ay' + By = f(t)$$

Finding a particular solution is easy if f is exponential:

$$f = f_0 e^{\alpha t}$$

If this is true, we can just guess a particular solution:

$$y_p = Ce^{\alpha t}$$

In physics, most of the important right hand sides are

1. e^{ax}, with $a < 0$
2. $\sin \omega x$, $\cos \omega x$, $e^{i\omega x}$
3. $e^{ax} \sin \omega x$
4. e^{ax}, with $a \in \mathbb{C}$

Rewriting our equation in operator terms:

$$(D^2 + AD + B)y = f$$

This operator is a polynomial $P(D)$ (polynomial in D). $P(D)$ is a linear operator on functions. It is also true that $P(D)e^{\alpha t} = P(\alpha)e^{\alpha t}$. This leads into the Exponential Input theorem:

Theorem 1.8. *If $f = e^{\alpha t}$, then*

$$y_p = \frac{e^{\alpha t}}{p(\alpha)} \qquad (p(\alpha) \neq 0)$$

Let's do an example of this:

$$y'' - y' + 2y = 10E^{-x} \sin x$$

Rewriting this in operator notation:

$$(D^2 - D + 2)y = 10e^{-x} \sin x$$

We will now complexify the equation:

$$(D^2 - D + 2)\tilde{y} = 10e^{-x}e^{ix}$$

We can then use the theorem to get the particular solution:

$$y_p = \frac{10e^{(-1+i)x}}{P(-1+i)} = \frac{10e^{(-1+i)x}}{(-1+i)^2 - (-1+i) + 2} = \frac{10e^{(-1+i)x}}{3 - 3i}$$

We can now take this and try to get the imaginary part:

$$\tilde{y}_p = \frac{10}{3} \frac{1}{1-i} \frac{1+i}{1+i} e^{-x}(\cos x + i \sin x)$$

Doing this out, we get

$$\tilde{y}_p = \frac{10}{6}(1+i)e^{-x}(\cos x + i\sin x)$$

Taking the imaginary part:

$$y_p = \frac{5}{3}e^{-x}(\cos x + \sin x) = \frac{5\sqrt{2}}{3}e^{-x}\cos\left(x - \frac{\pi}{4}\right)$$

Now lets look at the phase space of a damped driven oscillator, with $\omega \neq \omega_0$. If we have the equation $y'' + 4y' + 5y = \cos 3t$, we have previously seen that the complementary solution is $e^{-2t}(c_1\cos t + c_2 \sin t)$. We have that $P(D) = D^2 + 4D + 5$, and the complexified equation is

$$P(D)\tilde{y} = e^{3it}$$

We have that $\alpha = 3i$, so we can use the exponential input theorem:

$$P(3i) = -9 + 12i + 5 = 12i - 4 = 4(-1+3i)$$

This tells us the complex particular solution:

$$\tilde{y}_p = \frac{e^{3it}}{4(-1+3i)} = \frac{-1-3i}{40}e^{3it}$$

Taking the real part of this:

$$y_p = \frac{1}{40}(-\cos 3t + 3\sin 3t)$$

In phase space, that's an ellipse, time moving clockwise. Over time, the complementary solution decays away.

What if we have $P(\alpha) = 0$. This leads to the exponential shift rule:

$$P(D)(e^{\alpha x}u(x)) = e^{\alpha x}P(D+\alpha)u(x)$$

Suppose $P(D) = D$:

$$D(e^{\alpha x}u(x)) = (De^{\alpha x})u + e^{\alpha x}(Du) = \alpha e^{\alpha x}u + e^{\alpha x}(Su)$$

We can factor:

$$e^{\alpha x}(\alpha + D)u$$

Suppose we had $P(D) = D^2$:

$$D^2e^{\alpha x}u = D(D(e^{\alpha x}u)) = D(e^{\alpha x}(D+\alpha)u)$$

We can see that we can use the same rule again:

$$e^{\alpha x}(D+\alpha)((D+\alpha)u)$$

We could generalize this via induction, but the point is that once we have this shift rule, we can use it to solve for cases where α is a root of P.

What if $p(\alpha) = 0$? If we have the equation

$$(D^2 + AD + B)y = e^{\alpha x}$$

we have that $p(D) = D^2 + AD + B$. If we want $p(\alpha) = 0$, and it is a simple root of p:

$$y_p = \frac{te^{\alpha t}}{p'(\alpha)}$$

where $p'(\alpha)$ is the derivative of $p(\alpha)$ with respect to α.

If α is a double root, we have

$$y_p = \frac{t^2 e^{\alpha t}}{p''(\alpha)}$$

In general, if α is an nth root of p:

$$y_p = \frac{t^n e^{\alpha t}}{p^{(n)}(\alpha)}$$

Let's check this for a simple root when p is second order:

$$p(D) = (D - \beta)(D - \alpha) \quad \alpha \neq \beta$$

Let's compute $p'(D)$:

$$p'(D) = (D - \alpha) + (D - \beta)$$

Evaluating this at α:

$$p'(\alpha) = \alpha - \beta$$

Thus the particular solution that we're expecting is

$$y_p = \frac{te^{\alpha t}}{p'(\alpha)}$$

Now applying the operator $p(D)$ to the particular solution:

$$p(D)\frac{e^{\alpha t}}{p'(\alpha)} = \frac{1}{p'(\alpha)}p(D)e^{\alpha t}t$$

Now using the exponential shift rule:

$$= \frac{1}{p'(\alpha)}e^{\alpha t}p(D + \alpha)t$$

Now evaluating $p(D + \alpha)$:

$$p(D + \alpha) = (D + \alpha - \beta)(D + \alpha - \alpha) = D(D + \alpha - \beta)$$

This tells us that

$$\frac{1}{p'(\alpha)}e^{\alpha t}(D + \alpha - \beta)Dt$$

We know that $Dt = 1$, and then $(D + \alpha - \beta)1 = \alpha - \beta$:

$$y_p = \frac{1}{p'(\alpha)}e^{\alpha t}(\alpha - \beta)$$

Note that $p'(\alpha) = \alpha - \beta$, and thus we are left with

$$y_p = e^{\alpha t}$$

Thus we see that everything works out.

Let's do an example. If we have the ODE $y'' - 3y' + 2y = e^x$. We see that $\alpha = 1$, and $p(D) = (D^2 - 3D + 2) = (D-1)(D-2)$. Now computing $\frac{dp(D)}{dD}$:

$$p'(D) = 2D - 3$$

Plugging in $\alpha = 1$ and finding $p'(\alpha)$, we have that $p'(1) = -1$, giving us the solution

$$y_p = \frac{xe^x}{-1} = -xe^x$$

We can now find the solutions to the homogeneous equation, which we can see from the roots of the equation:

$$y_c = c_1 e^x + c_2 e^{2x}$$

Thus the general solution is

$$y = y_c + y_p = c_1 e^x + c_2 e^{2x} - xe^x$$

1.9 Resonance

Examples of these situations involve being on a swing, the Tacoma Narrows bridge, and opera singers breaking glasses by singing at the correct frequency. FM radio is also driven by resonance, and so is the way our ear interprets sound.

1.9.1 Undamped Resonance

Let's start with undamped resonance. Let's look at the undamped harmonic oscillator:

$$y'' + \omega_0^2 y = \cos(\omega t)$$

We can complexify the equation:

$$(D^2 + \omega_0^2)z = e^{i\omega t}$$

Using the Exponential Input Theorem:

$$z_p = \frac{e^{i\omega t}}{(i\omega)^2 + \omega_0^2} = \frac{e^{i\omega t}}{\omega_0^2 - \omega^2}$$

When we take the real part of this:

$$y_p = \frac{\cos \omega t}{\omega_0^2 - \omega^2}$$

If we plot the particular solution, we see that if $\omega < \omega_0$, we have a large amplitude, and when $\omega \ll \omega_0$, we have a small amplitude. Finally, what if $\omega > \omega_0$ by a small amount, we have a large amplitude that is opposite the amplitude of the driver, and if $\omega \gg \omega_0$, we have a small amplitude that is oppossite that of the driver.

In the case where $\omega = \omega_0$, we have that

$$(D^2 + \omega_0^2)z = e^{i\omega_0 t}$$

Thus the particular solution is

$$z_p = \frac{te^{i\omega_0 t}}{2i\omega_0}$$

Taking the real part of this:

$$y_p = \frac{t\sin\omega_0 t}{2\omega_0}$$

This is an linearly enveloped sinusoid (enveloped by $\frac{t}{2\omega_0}$ and $-\frac{t}{2\omega_0}$).

What do other particular solutions look like? We have that

$$y_p = \frac{\cos(\omega t)}{\omega_0^2 - \omega^2}$$

but we can also add anything that solves the homogeneous equation. If we add $\frac{-\cos(\omega_0 t)}{\omega_0^2-\omega^2}$:

$$y_p = \frac{\cos(\omega t) - \cos(\omega_0 t)}{\omega_0^2 - \omega^2}$$

In the limit where $\omega \to \omega_0$:

$$y_p = \frac{t\sin(\omega_0 t)}{2\omega_0}$$

1.9.2 Damped Resonance

For the damped resonance case, we have the equation

$$m\ddot{x} + c\dot{x} + kx = F(t)$$

When $F = 0$, we have the homogeneous equation, and we can guess $x = e^{rt}$ to get the characteristic equation:

$$r^2 + 2br + \omega_0^2 = 0$$

We can then use the quadratic formula:

$$r = -b \pm \sqrt{b^2 - \omega_0^2}$$

This gets us the pseudo-frequency, $\omega^2 = \omega_0^2 - b^2$. If we drive this system with a force, $f = \cos(\alpha t)$, what α gets the maximal amplitude? We find out that

$$\omega_R^2 = \omega_0^2 - 2b^2$$

2 Fourier Analysis

The point of Fourier Analysis is that every input can be written as a sum of exponentials, and we can then use the Exponential Input theorem. If we have the equation

$$y'' + Ay' + By = f(t) \quad f \neq 0$$

The EIT told us that sines, cosines, and exponentials make it easy to get the solution. Fourier analysis lets us take any input function and turn it into one of these forms. Any reasonable function $f(t)$ with period 2π can be written in the form

$$f(t) = c_0 + \sum_{n=1}^{\infty} a_n\cos(nt) + b_n\sin(nt)$$

Why do these Fourier Series solve the problem? Say we have the input $\cos(nt)$. We know that the response will be some function $y_n^c(t)$. If the input was $\sin(nt)$, we would have some particular solution $y_n^s(t)$. These are both particular solutions, and we can find them via the exponential input theorem. If we now built an input out of sums of these:

$$f(t) = c_0 + \sum_{n=1}^{\infty} a_n \cos(nt) + b_n \sin(nt)$$

We can use linearity, and we can apply the exponential input theorem term by term. We see that the terms $a_n \cos(nt)$ will return the response $a_n y_n^c(t)$, and similarly for the sine terms. We see that the particular solution that we get is

$$y_p = c_1 + \sum_{n=1}^{\infty} a_n y_n^c(t) + b_n y_n^s(t)$$

The reason we can do this is the superposition principle. The main process with Fourier is actually computing the series.

We are given some $f(t)$ with period 2π, and we want to get the values of a_n and b_n. We first start with orthogonality. If we have two functions u and v that are continuous or not very discontinuous on $\mathbb{R}$ with period 2π, then u and v are orthogonal (think perpendicular) if

$$\int_{-\pi}^{\pi} u(t)v(t)\,dt = 0$$

Theorem 2.1. *For any two of* $\sin(nt)$ *for* $n : 1 \ldots \infty$ *and* $\cos(mt)$ *for* $m : 0 \ldots \infty$*, they are orthogonal.*

Proof. To think about this, suppose we start with 2d vectors. If we have two vectors:

$$\vec{u} = u_1\hat{x} + u_2\hat{y} \qquad \vec{v} = v_1\hat{x} + v_2\hat{y}$$

If we take their inner product:

$$\vec{u} \cdot \vec{v} = u_1v_1 + u_2v_2$$

This is equal to 0 iff $\vec{u} \perp \vec{v}$.

Similarly, we can extend this to 3d vectors:

$$\vec{u} \cdot \vec{v} = u_1v_1 + u_2v_2 + u_3v_3$$

And once again this is 0 iff $\vec{u} \perp \vec{v}$.

Let us now jump to n-dimensional vectors:

$$\vec{u} = u_1\hat{e_1} + u_2\hat{e_2} + \cdots + u_n\hat{e_n} \qquad \vec{v} = v_1\hat{e_1} + v_2\hat{e_2} + \cdots + v_n\hat{e_n}$$

Note that just like unit vectors in 2 or 3 dimensions, $\hat{e_i} \cdot \hat{e_j} = 0$ if $i \neq j$ and 1 if $i = j$. The dot product is

$$\vec{u} \cdot \vec{v} = \sum_{i=1}^{n} u_iv_i = u_1v_1 + u_2v_2 + \ldots u_nv_n$$

We can now treat i as a continous number instead of a set of discrete points, which makes u and v into functions $u(t)$ and $v(t)$, and the sum becomes an integral:

$$\vec{u} \cdot \vec{v} = \int_{-\pi}^{\pi} u(t)v(t)\,dt$$

Another way of saying this is that functions are just infinite dimensional vectors.

Let's now prove that sines and cosines are orthogonal. There are several different methods of doing this, such as using trig identities, using complex exponentials, or this method, which generalizes more easily to other situations. This strategy is to use the ODE that these functions solve. These functions satisfy the equation

$$u_n'' + n^2 u_n = 0$$

This is a family of differential equations. We can prove the orthogonality case by case. We start by trying to prove that any two functions with $m \neq n$ are orthogonal:

$$\int_{-x}^{x} u'v'\,dt = 0$$

Integrating by parts:

$$\int_{-x}^{x} u'v'\,dt = (u'v)_{-\pi}^{\pi} - \int_{-\pi}^{\pi} u''v\,dt$$

We claim that the first term is always 0. Suppose v is a sine. Evaluated at π or $-\pi$, we get 0. Let's suppose that u is a cosine, which has a derivative of sine, giving us another 0. The only case we have to worry about is if both are cosines. Since cosine is even, evaluating them at $-\pi$ is equivalent to the evaluation at π, meaning that we get 0. Thus the first term is indeed always 0 if $m \neq n$. Let's now evaluate the rest of the integral. Using the ODE, we know that $u_n'' = -n^2 u_n$. Thus the integral becomes

$$n^2 \int_{-\pi}^{\pi} uv\,dt$$

Let's go back to the original integral, and let's integrate by parts the other way:

$$\int_{-\pi}^{\pi} u'v'\,dt = m^2 \int_{-\pi}^{\pi} uv\,dt$$

We know that 0 is equal to the integral minus itself:

$$0 = \int_{-\pi}^{\pi} u'v'\,dt - \int_{-\pi}^{\pi} u'v'\,dt = m^2 \int uv\,dt - n^2 \int uv\,dt = (m^2 - n^2) \int uv\,dt$$

Since we are working under the asummption that $m \neq n$, $m^2 - n^2 \neq 0$, and therefore the integral must be 0:

$$\int uv\,dt = 0$$

Thus the entire integral is equal to 0, which is what we want.

Let's now prove the case where $m = n$, and one is sine and the other is cosine. We have the integral

$$\int_{-\pi}^{\pi} \cos(nt)\sin(nt)\,dt = \int_{-\pi}^{\pi} \frac{1}{2}\sin(2nt)\,dt$$

When we integrate this, we get some cosine evaluated at $-\pi$ to π, and thus we have 0.

We have now proven that these functions are "vectors" that point in perpendicular directions. How do we calculate the lengths of these vectors? To get the length, we can compute them by taking the dot product with themselves:

$$\vec{u} \cdot \vec{u} = |u|^2$$

$$\int_{-\pi}^{\pi} \sin^2(nt) = \pi = \int_{-\pi}^{\pi} \cos^2(nt)\, dt$$

Thus these functions all have length squared of π. $\square$

We can now get on with converting an arbitrary function into an infinite sum of sines and cosines:

$$f = \cdots + a_k \cos(kt) + \cdots + a_n \cos(nt) + \cdots$$

Let's take the dot product of f and some vector $u = \cos(nt)$:

$$\int_{-\pi}^{\pi} f \cos(nt)\, dt = \int_{-\pi}^{\pi} \cdots + a_k \cos(kt)\cos(nt) + \cdots + a_n \cos^2(nt) + \cdots\, dt$$

By linearity of integration and orthogonality, we know that $\int a_k \cos(kt)\cos(nt) = 0$, and similarly for all other terms, except for the $\int a_n \cos^2(nt)$ term. We know that that term is $a_n\pi$:

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(nt)\, dt$$

Via the same reasoning, we know that

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(nt)\, dt$$

However, we still have the constant term to worry about:

$$f = c_0 + \cdots + a_k \cos(kt) + \cdots$$

We can think of the dot product with $\cos(0t)$, and once gain use orthogonality:

$$\int_{-\pi}^{\pi} f(t)\cos(0t)\, dt = \int_{-\pi}^{\pi} c_0\, dt = 2\pi c_0$$

Thus we have that $c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\, dt$.

Let's do an example. We have a function that alternates between positive and negative 1 every π. We want to calculate the Fourier coefficients, a_n and b_n. We can start with a_n:

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos(nt) f(t)\, dt$$

By inspection, we can see that this is equal to 0 by symmetry. Let's calculate b_n:

$$b_n = \frac{1}{\pi}\int_{-\pi}^{0} -1\sin(nt)\, dt + \frac{1}{\pi}\int_{0}^{\pi} \sin(nt)\, dt$$

$$b_n = \frac{2(1 - \cos(n\pi))}{n\pi}$$

Let's now think about what $\cos(n\pi)$ really is. This is -1 if n is odd, and $+1$ if n is even, which means that we have $\cos(n\pi) = (-1)^n$. This tells us that b_n is 0 when n is even, and $\frac{4}{n\pi}$ when n is odd. Thus we have that

$$f = \sum_{n\, odd}^{\infty} \frac{4}{n\pi}\sin(nt)$$

2.1 Parseval's Theorem

We now want to prove uniqueness of a Fourier series for a function. We want to prove that if $f(t) = g(t)$, then the corresponding Fourier series must be identical as well. We can compute the coefficients of both series:

$$a_n^f = \frac{1}{\pi}\int_{-\pi}^{\pi} f\cos(nt)\,dt = \frac{1}{\pi}\int_{-\pi}^{\pi} g\cos(nt)\,dt = a_n^g$$

And similarly for the b_n coefficients. Thus if two functions are equal, they have the same Fourier series. This does not tell us that if we have the same Fourier series they are both the same function. This leads us into Parseval's theorem, which tells us that the Fourier basis is complete. We want to show that the Fourier modes can be used to make any function. If we have two functions and their Fourier series:

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos(nt) + b_n\sin(nt) \qquad g(t) = \frac{\alpha_0}{2} + \sum_{n=1}^{\infty} \alpha_n\cos(nt) + \beta_n\sin(nt)$$

We can then compute the dot product $f \cdot g$:

$$f \cdot g = \int_{-\pi}^{\pi} f(t)g(t)\,dt = \int_{-\pi}^{\pi} \frac{a_0}{2}\left(\frac{\alpha_0}{2} + \sum_n \alpha_n\cos(nt) + \beta_n\sin(nt)\right) + \dots\,dt$$

Since we know that $\int_{-\pi}^{\pi}\cos^2(nt)\,dt = \pi$, and using orthogonality:

$$\int_{-\pi}^{\pi} fg\,dt = \frac{\pi}{2}a_0\alpha_0 + \cdots + a_n\alpha_n\pi + \cdots + b_m\beta_m\pi + \dots$$

This is Plancherel's theorem. This leads into Parseval's theorem:

Theorem 2.2.

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f^2\,dt = \left(\frac{1}{2}a_0\right)^2 + \frac{1}{2}\sum_{n=1}^{\infty} a_n^2 + b_n^2$$

2.2 Complex Formulation

We can express the Fourier series in terms of complex exponentials:

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos(nt) + b_n\sin(nt)$$

Using the fact that $\cos(x) = \frac{e^{ix}+e^{-ix}}{2}$ and $\sin(x) = \frac{e^{ix}-e^{-ix}}{2i}$, we can plug these into the expression for f, and we find that

$$f(t) = \frac{1}{2}a_0e^{i0t} + \frac{1}{2}\sum_{n=1}^{\infty}(a_n - ib_n)e^{int} + (a_n + ib_n)e^{-int}$$

This can be rewritten

$$f(t) = \sum_{n=-\infty}^{\infty} c_ne^{int}$$

We have found an equally valid basis for the infinite dimensional space of 2π periodic functions:

$$\{e^{int}, n \in \mathbb{Z}\}$$

2.3 Tips and Tricks

One of the major tricks that we can exploit is the symmetry of functions. A function is even if $f(-x) = f(x)$, and exhibits symmetry over the y-axis. A function is odd if $f(-x) = -f(x)$, and they exhibit 180 degree rotational symmetry.

If we integrate an even function in a symmetrical region, we can split the region in half and just double the half-integral:

$$\int_{-L}^{L} even\, dt = 2\int_{0}^{L} even\, dt$$

On the other hand, integrating an odd function over a symmetrical region is always 0:

$$\int_{-L}^{L} odd\, dt = 0$$

There are rules for odd/even function multiplication:

1. Even function $\times$ even function returns an even function
2. Odd function $\times$ odd function returns an even function
3. Even function $\times$ odd function returns an odd function

We know that $\cos(x)$ is an even function, and $\sin(x)$ is an odd function. We now want to show that even functions have cosine-only series, and that odd functions have sine-only series.

Suppose we have a function $f(t)$:

$$f(t) = \frac{a_0}{2} + \sum_n a_n \cos(nt) + b_n \sin(nt)$$

We can write the series for $f(-t)$:

$$f(-t) = \frac{a_0}{2} + \sum_n a_n \cos(-nt) + b_n \sin(-nt) = \frac{a_0}{2} + \sum_n a_n \cos(nt) - b_n \sin(nt)$$

Since we know that $f(t) = f(-t)$, we know that by uniqueness, the Fourier series must match, and thus the series must match term by term. We see that this implie that $b_n = -b_n$, and thus $b_n = 0$. Thus if f is even, then the Fourier series is cosine-only. Using similar logic, we find that if f is odd, the series is sine-only.

How can we leverage this symmetry to simplify our caclulations? We know that if $f(t)$ is even, then $f(t)\cos(nt)$ is also even. If we go to calculate a_n, we have that

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f\cos(nt)\, dt = \frac{2}{pi}\int_{0}^{\pi} f\cos(nt)\, dt$$

and if we calculate b_n:

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f\sin(nt)\, dt = 0$$

If f is odd:

$$a_n = 0$$

and

$$b_n = \frac{2}{\pi}\int_0^\pi f\sin(nt)\,dt$$

These symmetry arguments let us simplify our calculations a lot.

Let's do another example. If we have a 2π periodic function that is equal to t on the interval $(-\pi,\pi]$, we can find the Fourier series for this. We see that it is odd, and thus $a_n = 0$. We can then compute b_n:

$$b_n = \frac{2}{\pi}\int_0^\pi t\sin(nt)\,dt$$

Integrating by parts:

$$b_n = \frac{2}{\pi}\left[\left(\frac{-t\cos(nt)}{n}\right)_0^\pi + \int_0^\pi \frac{\cos(nt)}{n}\,dt\right] = \frac{2}{\pi}\left[\frac{-\pi\cos(n\pi)}{n} - 0 + \frac{1}{n}\int_0^\pi \cos(nt)\,dt\right]$$

$$b_n = \frac{-2\cos(n\pi)}{n}$$

Now using the fact that $\cos(n\pi) = (-1)^n$:

$$b_n = \frac{-2(-1)^n}{n}$$

This then gives us the final Fourier series:

$$f(t) = \sum_{n=1}^{\infty}\frac{-2(-1)^n}{n}\sin(nt)$$

If we want to compare the Taylor series expansion around $t = 0$ to the Fourier series, we can find the Taylor series:

$$f(t) = \sum_{n=0}^{\infty}\frac{t^n f^{(n)}(0)}{n!} = 0 + t + 0 + 0 + \dots$$

Thus our Taylor expansion around $t = 0$ is just $f(t) = t$. We see that this completely breaks down at any discontinuities. The Fourier series doesn't have this issue.

Theorem 2.3. *If f is continuous at t_0, then $f(t_0)$ is equal to the Fourier series at t_0. If f has a jump discontinuity then the Fourier series converges to the midpoint of the jump.*

These are the Dirichlet conditions.

We also notice something known as the Gibb's phenonemon, which is what happens to the series as it approaches a jump discontinuity, where we see a large spike right next to the discontinuity.

Now let's look at functions that are periodic over some general period $2L$. We have shown the case where $L = \pi$, but we now want to find it for some general L. We want to find the "natural" functions for generating Fourier series on a general period. If we have a function of time, and it has a period of $2L$:

$$f(t) = f(t + 2L)$$

We know that $\cos\left(\frac{n\pi t}{L}\right)$, with $n \in \mathbb{Z}^+, 0$ is also $2L$-periodic. Similarly, $\sin\left(\frac{n\pi t}{L}\right)$ is also $2L$-periodic. Thus we see that $e^{i\left(\frac{n\pi t}{L}\right)}$ works the same way, because it is made up of the two $2L$ periodic functions that we have. These are the "natural" functions to use when we have an arbitrary period of $2L$.

Let's say we start with a function of a variable u. If changing from u to t maps 0 to 0 and π to L, we have to do $t = L\frac{u}{\pi}$, which is equivalent to $u = \frac{t\pi}{L}$. We can now write a Fourier series for a function that is periodic with $2L$:

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi t}{L}\right) + b_n \sin\left(\frac{n\pi t}{L}\right)$$

We also need to change the definitions of a_n and b_n:

$$a_n = \frac{1}{L}\int_{-L}^{L} f(t)\cos\left(\frac{n\pi t}{L}\right)\,dt \qquad b_n = \frac{1}{L}\int_{-L}^{L} f(t)\sin\left(\frac{n\pi t}{L}\right)\,dt$$

If we replace π with L, and use the natural functions instead, we can convert the formula with period 2π into a formula for a period of $2L$.

Suppose we have a function that is defined on just $[0, L]$, such as $f(t) = t^2$. We can make what's known as a periodic extension, and copy/paste it every L. We can do either an even periodic extension or an odd periodic extension. The reason this is useful is because the even PE has just a pure cosine series, and the odd PE has a pure sine series.

2.4 Discrete Fourier Transforms

The DFT is useful when we look at real world data. For example, if we have some experiment with a signal. Data in the real world is often stored discretely, not continuously. We can store data as a list of coordinates:

$$(0, f(0)), (\Delta t, f(\Delta t)) \ldots ((n-1)\Delta t, f((n-1)\Delta t))$$

We have n datapoints. Conventionally, we pick a set of units where $\Delta t = 1$, so our samples take place at integer locations. What are the natural functions for describing this kind of data? We can create a periodic extension of a set of datapoints just by repeating the list of points again and again. The period of the list is just the number of elements in the dataset, which is N. We can find that the "natural functions" are vectors

$$\vec{u}_n = e^{\frac{2\pi i x n}{N}}$$

where $\vec{u}_n$ is a vector with N elements, each one of which corresponds to a value of x.

If we have a list of data f_x:

$$f_x = \sum_{n=0}^{N-1} F_n e^{\frac{2\pi i x n}{N}}$$

where x runs from 0 to $N-1$, and n also runs from 0 to $N-1$.

If we compute the dot product of two of these natural functions:

$$\vec{u}_n \cdot \vec{v}_n = \sum_x u_n^* v_m = \sum_x e^{-\frac{2\pi i x n}{N}} e^{\frac{2\pi i x m}{N}}$$

$$= \sum_x e^{\frac{2\pi i (m-n)x}{N}}$$

When $m = n$, we have $e^0 = 1$, so we are left with N. When $m \neq n$, we get 0, (prove this on homework 5). Essentially, we have that

$$\vec{u}_n \cdot \vec{v}_n = N\delta_{nm}$$

We then have the Discrete Fourier Transform:

$$f_x = \sum_k F_k e^{\frac{2\pi i}{N}kx}$$

How do we find F_k? To compute it, we use orthogonality:

$$\sum_n e^{-\frac{2\pi i}{N}jn} f_n = \sum_n e^{-\frac{2\pi i}{N}jn} \sum_k F_k e^{\frac{2\pi i}{N}kn} = \sum_k F_k \sum_n e^{\frac{2\pi i}{N}(k-j)n}$$

Using the argument we just made, we know that the sum over n is just $N\delta_{kj}$:

$$\sum_k F_k N \delta_{kj} = NF_j$$

If we now put a $\frac{1}{N}$ before every step, we have that

$$F_j = \frac{1}{N}\sum_n e^{-\frac{2\pi i}{N}jn} f_n$$

We can look into the Fourier Modes forming a basis. We can write out that

$$f_n = \sum_x f_x \delta_{xn}$$

f_n is a vector in an N-dimensional space, and for some fixed n, δ_{nx} is the 0 vector with a single 1 in the xth slot. We have a simple orthonormal basis:

$$\hat{e}_n \cdot \hat{e}_m = \sum_x \delta_{nx}\delta_{mx} = \delta_{mn}$$

F_k describes the same vector as f_n, but in a different basis. For a fixed n, u_n is a vector of N objects, with the entries:

$$u_{nk} = e^{\frac{2\pi i}{N}nk}$$

These vectors form an orthogonal basis (not orthonormal since the dot product $u_n \cdot u_m = \sum_x u^*_{nx} u_{mx} = N\delta_{mn}$ is not 1).

We have that

$$f_n = \sum_k F_k e^{\frac{2\pi i}{N}nk}$$

Where

$$F_k = \frac{1}{N}\sum_n f_n e^{-\frac{2\pi i}{N}nk}$$

F is the Discrete Fourier Transform (DFT) of f.

There are certain conventions, such as placing the negative and positive signs in different places, as well as a convention where we change where the $\frac{1}{N}$ is placed.

Let's do an example. If we have that $f = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \end{bmatrix}$, we want to find F. We see that $N = 4$. Let's start by computing F_0:

$$F_0 = \frac{1}{4}\sum_n f_n e^0 = \frac{1}{4}[1+2+1+2] = \frac{3}{2}$$

Now computing F_1:

$$\frac{1}{4}\left[(-i)^0 \cdot 1 + (-i)^1 \cdot 2 + (-i)^2 \cdot 1 + (-i)^3 \cdot 2\right] = 0$$

If we do this process similarly for the other two terms, we find that

$$F = \frac{1}{4}\begin{bmatrix} 6 \\ 0 \\ -1 \\ 0 \end{bmatrix}$$

If we have real f_n, the output of the DFT will give us N real numbers. If F_k is complex, then we will end up with $2N$ real numbers. We ideally want to just end up with N, so how do we resolve this? If f is real, we have that F is even symmetric, where $F_k = F^*_{-k}$. This brings down the amount we have to specify by half, since we can just pair things up on both ends, giving us N real numbers. Thus we have the same amount of information in F that we do in f.

2.5 Dirac Delta Function

Let us compare the Kronecker delta function to the Dirac delta. The Kronecker delta takes in two indices/arguments, m and n, and it is defined to be 1 when $m = n$ and 0 otherwise. We have seen one of the uses of this function, for picking out a piece of a vector or sum, as a sort of indicator function. If we have a list or vector f_x, getting a particular value of the list f_n can be obtained by summing over all the values of x and multiply by the Kronecker:

$$f_n = \sum_x \delta_{nx} f_x$$

The Dirac delta function is similar, but is a function of two continuous variables instead:

$$f(y) = \int_{-\infty}^{\infty} \delta(x-y) f(x)\, dx$$

The Dirac Delta picks out $f(y)$ from the integral, and nothing else. We want the following to be true:

$$1 = \int_{-\infty}^{\infty} \delta(x-y)\, dx$$

To make this true, we need the value when $x = y$ to be ∞. This function only really makes sense when it's inside of an integral. Essentially, the Dirac delta is the limit of a sequence of functions. We're looking for a function that when integrated gives us 1. If we look at when $y = 0$:

$$\int_{-\infty}^{\infty} \delta(x)\, dx = 1$$

We also want

$$\int_{-\infty}^{\infty} \delta(x) f(x)\, dx = f(0)$$

If we think back to the Gaussian:

$$g_\sigma(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}}$$

The coefficient on the outside was chosen to make sure that

$$\int_{-\infty}^{\infty} g_\sigma(x)\, dx = 1$$

If we plot the Gaussian centered around $x = 0$, the larger the value of σ, the wider the distribution is. As we decrease σ, the distribution gets taller and narrower. If we take σ to be very small, it falls off very quickly, and gets taller and taller. The delta function is the limit as $\sigma \to 0$ of the Gaussian:

$$\delta(x) = \lim_{\sigma \to 0} g_\sigma(x)$$

There are all sorts of other functions that have the same behavior, such as the unit impulse that we looked at in homework 6. Let's do an example. If we have an infinitely thin, very long rod, and we place a point particle at some x_0, with charge Q. If we want to compute the total amount of charge to the left of some x value:

$$Q(x) = \int_{-\infty}^{x} q(y)\,dy$$

Since we put down a point particle, $Q(x)$ for all values to the left of the point particle is 0, and to the right all points have $Q(x) = Q$. What is the charge density q? It's the Dirac Delta:

$$q = Q\delta(x - x_0)$$

Let's make sure:

$$\int_{-\infty}^{x} Q\delta(y - x_0)\,dy = Q\int_{-\infty}^{x} \delta(y - x_0)\,dy$$

We can see that this is 0 if $x < x_0$, and Q if $x > x_0$.

Suppose we had some continuous function $f(t)$ that we wanted to sample at certain times $\{n\}$. We can sample it with the delta function:

$$f(n) = \int_{-\infty}^{\infty} \delta(t - n)f(t)\,dt$$

We can actually relate the step function to the delta function:

$$\theta'(x) = \delta(x)$$

where $\theta(x) = 1$ for $x > 0$ and $\theta(x) = 0$ for $x < 0$.

Now let's look at the derivatives of the delta function.

$$\int_{-\infty}^{\infty} \delta'(x)f(x)\,dx$$

Integrating by parts, with $dv = \delta'(x)$ and $u = f(x)$:

$$[f(x)\delta(x)]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \delta(x)f'(x)\,dx$$

The first term is just 0, and the second term is just $f'(0)$:

$$\int_{-\infty}^{\infty} \delta'(x)f(x)\,dx = -f'(0)$$

What about scalars inside the delta function:

$$\delta(ax)$$

Remember that

$$\delta(x) = \lim_{\sigma\to 0} g_\sigma(x) \qquad g_\sigma(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}}$$

If we now plug in ax for x:

$$g_\sigma(ax) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{a^2x^2}{2\sigma^2}}$$

Let us now absorb the a into σ. Let $s = \frac{\sigma}{a}$, which means that $\sigma = as$:

$$\lim_{\sigma\to 0} = \lim_{s\to 0}$$

We can then rewrite the rest of it:

$$g_\sigma(ax) = \frac{1}{\sqrt{2\pi s^2 a^2}} e^{-\frac{x^2}{2s^2}} = \frac{1}{\sqrt{a^2}} g_s(x)$$

Thus we have that

$$\lim_{\sigma\to 0} g_\sigma(ax) = \lim_{s\to 0} \frac{1}{|a|} g_s(x) = \frac{1}{|a|}\delta(x)$$

What about the delta function of another function, $\delta(g(x))$. If g has no zeros, then the delta function is always 0. Suppose g has zeroes x_i. In the neighborhood of each x_i, g looks linear, with slope $g'(x_i)$. We can Taylor expand in the neighborhood of x_i:

$$\delta(g(x)) = \delta(0 + g'(x_i)(x - x_i) + \dots) = \delta(g'(x_i)(x - x_i)) = \frac{\delta(x - x_i)}{|g'(x_i)|}$$

We can then sum over all x_is:

$$\delta(g(x)) = \sum_i \frac{\delta(x - x_i)}{|g'(x_i)|}$$

Lets do an example. If we have $\delta(x^2 - a^2)$, we have that $g(x) = x^2 - a^2 = (x - a)(x + a)$. We see that we have zeroes at $x = \pm a$. Taking the derivative:

$$g'(x) = 2x$$

We can then write out the delta function:

$$\delta(x^2 - a^2) = \frac{1}{|2a|}\delta(x - a) + \frac{1}{|-2a|}\delta(x + a) = \frac{1}{|2a|}\left(\delta(x - a) + \delta(x + a)\right)$$

If we now put this inside an integral:

$$\int \delta(x^2 - a^2) f(x)\, dx$$

If we look at when $x \approx -a$, such as $x = -a + \epsilon$. The delta function will output:

$$\delta((-a + \epsilon - a)(-a + \epsilon + a)) = \delta(-2a\epsilon + \epsilon^2)$$

We know that ϵ is very small, so we can drop the ϵ^2 term, giving us $\delta(-2a\epsilon)$.

2.6 Fourier Transforms

If we have an operator, like the differential operator, it takes in a function of x and spits out another function of x. A transformation is similar, in that it takes in a function of x, but it spits out a function that is in a different variable.

Let's look at the complex exponential Fourier series. If we have a function $f(x)$ over a generic period:

$$c_n = \frac{1}{2L}\int_{-L}^{L} f(x)e^{\frac{-in\pi x}{L}}\,dx$$

And

$$f(x) = \sum_n c_n e^{\frac{in\pi x}{L}}$$

This takes in a function of x, and spits out a coefficient, but the Fourier transform spits out another function.

The Fourier Transform of a function $f(x)$ is defined as

$$F(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikx} f(x)\,dx$$

These are like the c_ns that we got when we did Fourier Series. We can similarly reconstruct $f(x)$:

$$f(x) = \int_{-\infty}^{\infty} F(k)e^{ikx}\,dk$$

Let's write $f(x)$ out:

$$\begin{aligned} f(x) &= \int_{-\infty}^{\infty} dk\, e^{ikx} \cdot \frac{1}{2\pi}\int_{-\infty}^{\infty} dy\, e^{-iky} f(y) \\ &= \int_{-\infty}^{\infty} dy\, f(y) \frac{1}{2\pi}\int_{\infty}^{\infty} e^{ik(x-y)} \end{aligned}$$

This inner integral is just $\delta(x-y)$, as we will show. To do this, we want to carefully let the period of the Fourier series go to infinity. We can write down c_n:

$$c_n = \frac{1}{2L}\int_{-L}^{L} f(x)e^{-i\frac{\pi n}{L}x}\,dx$$

Let $k_n = \frac{n\pi}{L}$. The change between each k $\Delta k_n = \frac{\pi}{L}$, and when $L \to \infty$, $\Delta k_n \to 0$. We can substitute in k_n:

$$c_n = \frac{\Delta k_n}{2\pi}\int_{-L}^{L} f(x)e^{-ik_n x}\,dx$$

Plugging these into the Fourier series:

$$f(y) = \sum_n \left[\frac{\Delta k_n}{2\pi}\int_{-L}^{L} e^{-k_n x} f(x)\,dx\right] e^{ik_n y}$$

If we say that $F(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{-ik_n x}\,dx$, then

Now as we take $L \to \infty$, we see that we get $F(k_n)$ inside the sum, and we are left with

$$f(y) = \sum_n \Delta k_n F(k_n) e^{ik_n y}$$

The pieces that are left have that $\Delta k_n \to 0$, meaning that this is just a Riemann sum, giving us an integral:

$$f(y) = \int F(k)e^{iky}$$

This is just the Fourier Transform.

If we look at the formulas for $F(k)$ and $f(x)$ side by side, we see that they only have superficial differences, one being a function of k, the other of x, on having a $\frac{1}{2\pi}$, and a sign difference in the exponential. People say that k and x are "dual", because they store the same amount of information. Also note that their units/dimensions are related. kx is dimensionless, and thus if x is length, k must be one over length. Another thing, if we Fourier transform a function of time, we get that $k = \omega$, with units of one over time, which is frequency.

There are many conventions when talking about Fourier Transforms, such as the placement of the sign, the placement of the $\frac{1}{2\pi}$, and sometimes the FT of $f(x)$ (what we write as $F(k)$) is written as $f(k)$. We have essentially overloaded the definition of f, to be two different functions.

2.7 Sine and Cosine Transforms

Let's say we want to take the sine transform of $f_s(x)$, which spits out $F_s(k)$:

$$f_s(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_s(kx)\, dk$$

$$F_s(k) = \sqrt{\frac{2}{\pi}} \int_0^\infty f_s(x) \sin(kx)\, dx$$

For the cosine transform, we just change the sines to cosines.

2.8 Examples of Fourier Transforms

We can take the Fourier transform of the delta function:

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \delta(x) e^{-ikx}\, dx = \frac{1}{2\pi}$$

The Fourier transform of $\delta(x - y)$:

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \delta(x - y) e^{-ikx}\, dx = \frac{1}{2\pi} e^{-iky}$$

If we want to find the inverse Fourier transform of $\frac{1}{2\pi}$:

$$f(x) = \int_{-\infty}^{\infty} \frac{1}{2\pi} e^{ikx}\, dk = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx}\, dk = \delta(x)$$

If we want to find the inverse Fourier transform of $\frac{1}{2\pi} e^{-iky}$:

$$f(x) = \int_{-\infty}^{\infty} \frac{1}{2\pi} e^{-iky} e^{ikx}\, dk = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ik(x-y)}\, dk = \delta(x - y)$$

Let's do the FT of e^{ipx}:

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ipx} e^{-ikx}\, dx = \frac{1}{2\pi} \int e^{ix(p-k)}\, dx = \delta(p - k)$$

Taking the Fourier transform of a constant function, $f(x) = 1$:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} 1e^{-ikx}\,dx = \delta(k)$$

If we Fourier transform this again, we have already found that we get $\frac{1}{2\pi}$. With our convention, taking the FT twice is the same as dividing by 2π, and this is true for all functions.

2.9 Linearity

If we want to find the Fourier transform of $f(x) + g(x)$:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}(f(x) + g(x))e^{-ikx}\,dx$$

Integration follows linearity, so we can easily split this up, and we see that we have

$$F(k) + G(k)$$

Now if we look at scaling a function $cf(x)$, we see that the scalar will just pull out of the integral, and thus we will be left with $cF(k)$. Thus the Fourier transform is a linear transformation.

We can also look at Fourier transforming the derivative of a function:

$$\frac{1}{2\pi}\int \frac{df}{dx}e^{-ikx}\,dx$$

Integrating by parts:

$$\frac{1}{2\pi}\left[\left[e^{-ikx}f\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f(-ik)e^{-ikx}\,dx\right] = 0 + \frac{ik}{2\pi}\int fe^{-ikx}\,dx = ikF(k)$$

Where we have made the assumption that f goes to 0 as $x \to \infty$. Since we can "recursively" define the derivative:

$$\frac{d^n f}{dx^n} = \frac{d}{dx}\left(\frac{d^{n-1}f}{dx^{n-1}}\right)$$

We know that the Fourier transform of the nth derivative is

$$\frac{d^n f}{dx^n} \to (ik)^n F(k)$$

2.10 Solving ODEs with Transforms

Say we have the equation

$$\ddot{y} + 2b\dot{y} + \omega_0^2 y = f(t)$$

We can Fourier transform both sides:

$$i^2\omega^2 Y(\omega) + 2bi\omega Y(\omega) + \omega_0^2 Y(\omega) = F(\omega)$$

$$Y(\omega) = \frac{F(\omega)}{\omega_0^2 - \omega^2 + 2ib\omega}$$

We now know the Fourier transform of the solution, and we can take the inverse Fourier transform to get $y(t)$.

3 Laplace Transforms

Laplace Transforms help solve IVPs, where we have some system governed by an equation, as well as initial conditions that can define the constants. The idea behind Laplace transforms is that it is hard to go from the equation to a system.

The methodology is to take the equation and to take the Laplace Transform of it. This returns some algebraic equation in terms of $Y = \mathcal{L}(y)$, which can be solved for Y, and then do an inverse Laplace transform to get y. The hardest part of this process is taking the inverse Laplace transform of the function Y.

Let's start with power series. Power series let us express some function $A(x)$:

$$A(x) = \sum_{n=0}^{\infty} a_n x^n = \sum a(n)x^n$$

Let's do an example. If all of the values of a_n are 1:

$$A(x) = 1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}$$

We can do another example, where $a(n) = \frac{1}{n!}$:

$$A(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = e^x$$

The Laplace transform is the continuous analogue of the power series, where n is no longer just integers.

$$A(x) = \int_0^{\infty} a(t)x^t\,dt$$

This is not the way people normally write it. Instead, it is generally written:

$$A(x) = \int_0^{\infty} a(t)e^{t\ln x}\,dt$$

Note that this is unlikely to converge if $\ln x > 0$ or $x > 1$. It is really bad if $x < 0$. Thus we want $0 < x < 1$. We can do a substitution $-s = \ln x$, to write the transform as

$$A(s) = \int_0^{\infty} a(t)e^{-st}\,dt$$

The Laplace transform takes in a function of t, and spits out a function of s. There are multiple notations for representing it, but often it takes the form

$$\mathcal{L}(f(t)) = F(s)$$

Note that the transform is linear, so $\mathcal{L}(f+g) = \mathcal{L}(f) + \mathcal{L}(g)$, and $\mathcal{L}(cf) = c\mathcal{L}(f)$.

Let's do some examples. Lets take the function $f(t) = 0$. As expected, $\mathcal{L}(0) = 0$. Let's do the Laplace transform of 1.

$$\mathcal{L}(1) = \int_0^{\infty} 1e^{-st}\,dt = \lim_{R\to\infty}\int_0^{R} e^{-st}\,dt = \lim_{R\to\infty}\left[\frac{e^{-st}}{-s}\right]_0^R$$

Evaluating the bounds and computing the limit:

$$= \lim_{R\to\infty} \frac{e^{-sR}-1}{s} = \frac{1}{s}$$

However, this is only true if $s > 0$, and remember that $s = -\ln x$, with $0 < x < 1$.

Let's look at the Laplace transform of $e^{at}f(t)$:

$$\mathcal{L}(e^{at}f(t)) = \int_0^\infty e^{-st}(e^{at}f(t))\,dt$$

$$= \int_0^\infty e^{-t(s-a)}f(t)\,dt = F(s-a)$$

We see that it is the Laplace transform of $f(t)$ evaluated at a.

We can compute the Laplace transform of the exponential function itself, $f(t) = e^{at}$. We can rewrite this as $e^{at} \times 1$. Using what we just found about and the Laplace transform of 1, we can deduce that the Laplace transform of e^{at} is

$$\mathcal{L}(e^{at}) = \frac{1}{s-a} \quad (s > a)$$

This is the exponential shift rule.

We can do another one. Take the function $f(t) = e^{(a+ib)t}$. Using the same trick as for the regular exponential, we find that

$$\mathcal{L}(e^{(a+ib)t}) = \frac{1}{s-(a+ib)} \quad (s > a)$$

We can do the transform of $\cos(at)$, by using the fact that

$$\cos(at) = \frac{1}{2}(e^{iat} + e^{-iat})$$

Using linearity and the exponential shift rule:

$$\mathcal{L}(\cos(at)) = \frac{1}{2}\left(\frac{1}{s-ia} + \frac{1}{s+ia}\right)$$

We can see that the complex conjugate of this is just itself, and thus the function must be real:

$$\mathcal{L}(\cos(at)) = \frac{1}{2}\frac{2s}{s^2+a^2} = \frac{s}{s^2+a^2} \quad (s > 0)$$

We can find an inverse of a Laplace Transform, but its horrible:

$$f(t) = \mathcal{L}^{-1}(F(s)) = \frac{1}{2\pi i}\lim_{T\to\infty}\int_{\gamma-iT}^{\gamma+iT} e^{st}F(s)\,dt$$

Instead of doing this, we'll build a table.

Let's do an example. If we have $F(s) = \frac{1}{s(s+3)}$, and we want to find $f(t)$. We can do partial fraction decomposition first:

$$\frac{1}{s(s+3)} = \frac{A}{s} + \frac{B}{s+3}$$

Where

$$A(s+3)+Bs=1 \to 3a+(a+b)s=1$$

Now letting $s=0$, we find that $a=\frac{1}{3}$, and from there we see that $b=-\frac{1}{3}$. We now have that

$$F(s)=\frac{1}{3s}-\frac{1}{3s+9}$$

Since the Laplace transform is linear, we can just split the two and compute the inverses separately:

$$f(t)=\frac{1}{3}-\frac{1}{3}e^{-3t}$$

There is an ambiguity in the inverse Laplace transform, because the inverse doesn't care about anything before 0, so different functions that match after 0 can have the same transform.

Let's do polynomials, such as $f(t)=t^n$:

$$\int_0^\infty e^{-st}t^n\,dt$$

Integrating by parts:

$$=\left[\frac{t^n e^{-st}}{-s}\right]_0^\infty-\int_0^\infty nt^{n-1}\frac{e^{-st}}{-s}\,dt=\frac{n}{s}\int_0^\infty t^{n-1}e^{-st}\,dt$$

We have shown that $\mathcal{L}(t^n)=\frac{n}{s}\mathcal{L}(t^{n-1})$. Now recursively using this rule, and we find that $\frac{n!}{s^{n+1}}$.

Now taking the Laplace transform of the step function $\mathcal{L}(u_{ab}(t))$:

$$u_{ab}(t)=u_a(t)-u_b(t)=u_(t-a)-u(t-b)$$

$$\mathcal{L}(u_{ab}(t))=\frac{e^{-as}}{s}-\frac{e^{-bs}}{s}$$

Taking the Laplace transform of an impulse of width w and height h:

$$\mathcal{L}(hu_{0w}(t))=h\left(\frac{e^0}{s}-\frac{e^{-ws}}{s}\right)=\frac{h}{s}-\frac{he^{-ws}}{s}$$

If we take the limit as $w\to 0$ when $h=\frac{1}{w}$, we expect the impulse to approach the delta function. Taking the limit of the Laplace transform of the impulse with $h=\frac{1}{w}$:

$$\lim_{w\to 0}\frac{1-e^{-ws}}{ws}=\lim_{w\to 0}\frac{se^{-ws}}{s}=\lim_{w\to 0}e^{-ws}=1$$

Thus we have shown that the Laplace transform of the Dirac Delta is 1.

Let's do an example diffeq:

$$y''+y=A\delta(t-\frac{\pi}{2})$$

With $y(0)=1$ and $y'(0)=0$. Taking the transform of the equation:

$$s^2Y-sy(0)-y'(0)+Y=Ae^{-\frac{\pi}{2}s}$$

Solving for Y:

$$Y = \frac{s + Ae^{-\frac{\pi}{2}s}}{s^2+1} = \frac{s}{s^2+1} + \frac{Ae^{-\frac{\pi}{2}s}}{s^2+1}$$

Now taking the inverse Laplace Transform:

$$y = \cos t + Au\left(t - \frac{\pi}{2}\right)\sin\left(t - \frac{\pi}{2}\right)$$

We can do another example:

$$y' + ay' + by = f(t)$$

with $y(0) = 0$ and $y'(0) = 0$. Taking the Laplace transform:

$$s^2Y + asY + bY = F(s)$$

Rewriting:

$$(s^2 + as + b)Y = F$$
$$Y = \frac{F}{s^2 + as + b}$$

$$\frac{1}{s^2 + as + b} = W(s)$$

This is known as the transfer function, and its inverse Laplace transform $W(t)$ is known as the Green's Function. It is the response of the system to $f = \delta$.

3.1 Convolution

If we have two functions $F(x) = \sum_n a_n x^n$ and $G(x) = \sum_n b_n x^n$, with Laplace transforms $F(s) = \int_0^\infty e^{-st} f(t)\, dt$ and $G(s) = \int_0^\infty e^{-st} g(t)\, dt$. Suppose we want to find the Laplace transform of $f(t)g(t)$? If we look at the power series:

$$f(t)g(t) = \sum_n a_n b_n x^n$$

We can write this as sum new power series

$$F \cdot G = \sum_n c_n x^n$$

The constant term will just be $a_0 b_0$ in this series. For c_1, we want all the terms that have x^1 in them. The only way to get them is if we multiply a constant against a linear term:

$$c_1 = a_0 b_1 + a_1 b_0$$

For c_2:

$$c_2 = a_0 b_2 + a_1 b_1 + a_2 b_0$$

In general:

$$c_n = \sum_{j=0}^{n} a_j b_{n-j}$$

From this, we can get the convolution operator:

$$(f * g)(t) = \int_0^t f(\tau)g(t-\tau)d\tau$$

Another definition is

$$F(s)G(s) = \int_0^\infty e^{-st}(f*g)\,dt$$

We can find this via writing out the integrals on the left side:

$$\int_0^\infty e^{-su}f(u)\,du \int_0^\infty e^{-sv}g(v)\,dv$$

Writing this as a double integral:

$$\iint_0^\infty e^{-s(u+v)}f(u)g(v)\,dudv$$

$$= \int_0^\infty dt \int_0^t du\, e^{-st}f(u)g(t-u)\cdot J$$

Where J is the Jacobian, which in this case is just 1. the e^{-st} can be pulled out to the second integral:

$$= \int_0^\infty dte^{-st}\int_0^t du\, f(u)g(t-u)$$

We see that the inner integral is what we defined the convolution to be.

Essentially the convolution tells us that

$$\mathcal{L}(f*g) = \mathcal{L}(f)\cdot\mathcal{L}(g) = \mathcal{L}(g*f)$$

Let's do some examples. If we want to find $t^2 * t$:

$$\int_0^t d\tau\,(\tau^2)(t-\tau) = \int_0^t d\tau\,(t\tau^2-\tau^3) = \left[\frac{t\tau^3}{3}\right]_0^t - \left[\frac{\tau^4}{4}\right]_0^t = \frac{t^4}{12}$$

Let's take the Laplace transform of this function:

$$\mathcal{L}\left(\frac{t^4}{12}\right) = \frac{2}{s^5}$$

Recall that $\mathcal{L}(t^n) = \frac{n!}{s^{n+1}}$. Taking the transform of the first and second pieces:

$$\mathcal{L}(t^2) = \frac{2!}{s^3} \quad \mathcal{L}(t) = \frac{1}{s^2}$$

Multiplying them together, we see that $\frac{2}{s^3}\frac{1}{s^2} = \frac{2}{s^5}$, which is what the convolution told us!

Computing the convolution of $f(t)$ and 1, $f(t)*1$:

$$= \int_0^t d\tau\, f(\tau) = \int_0^t f(\tau)\,d\tau$$

Let us now convolve 1 with f:

$$= \int_0^t d\tau\, 1\cdot f(t-\tau)$$

Doing a change of variables, and making $u = t - \tau$:

$$\int_t^0 -du\, f(u) = \int_0^t du\, f(u)$$

Thus we have shown that the convolution in this case is commutative, $(f(t) * 1) = (1 * f(t))$.

Another interesting thing is that the δ function is like the identity for the convolution operator:

$$f * \delta = \int_0^t d\tau\, f(\tau)\delta(t - \tau) = f(t)$$

We can also do this via Laplace transforms. Let $F = \mathcal{L}(f)$. We know that $\mathcal{L}(\delta) = 1$, and thus $F \cdot 1 = F$.

Lets talk about properties of the convolution. One property is that it is a bilinear operator. This means that

$$(f + g) * h = f * h + g * h$$

And

$$f * (g + h) = f * g + f * h$$

Looking at the interaction with scalars:

$$(cf) * (dg) = cd(f * g)$$

Something that is harder to show, is that convolution is associative:

$$(f * g) * h = f * (g * h)$$

Let's do an example. If every week we deposit some nuclear waste, starting at $\tau = 0$. At time t, how much nuclear waste is in the facility can be given via:

$$\sum_{\tau_i} f(\tau_i)\Delta\tau_i e^{-k(t-\tau_i)}$$

Converting this to a continuum version, we have

$$f * e^{-kt}$$

Moving back to ODES with the Green's Function:

$$y'' + ay' + by = f \qquad y_0 = 0, y_0' = 0$$

Taking the Laplace transform:

$$s^2Y + asY + bY = F \rightarrow Y = \frac{F}{s^2 + as + b} = \frac{1}{s^2 + as + b} \cdot F$$

Each of these is the Laplace transform of a function, the first being the weight $W(t)$, convolved with $f(t)$:

$$y = W(t) * f$$

If we have two functions f and g, with Fourier transforms F and G, we have that $\mathcal{F}\left(\frac{f*g}{2\pi}\right) = FG$:

$$F(k)G(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iku} f(u)\,du \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikv} g(v)\,dv$$

$$\frac{1}{4\pi^2}\iint du\,dv\,e^{-i(u+v)k} f(u)g(v)$$

Now doing a change of variables, with $x = u + v$, making $v = x - u$ and $dv = dx$. The Jacobian of this transformation is 1, so we are left with

$$\frac{1}{4\pi^2}\iint du\,dx\,e^{-ikx} f(u)g(x-u)$$

Rewriting this:

$$\frac{1}{2\pi} dx\,e^{-ikx} \frac{1}{2\pi}\int du\,f(u)g(x-u)$$

This is a Fourier transform:

$$= \mathcal{F}\left(\frac{f*g}{2\pi}\right)$$

Thus proving the identity we stated.

4 Linear Spaces and Eigenvalue Problems

4.1 Linear Spaces

If you have an arbitrary vector $\vec{f}$, we can break it down into its components given the basis vectors of your basis:

$$\vec{f} = f_x\hat{x} + f_y\hat{y}$$

Suppose we have some vector and we want to find its component along a certain direction. We can obtain it via projection:

$$f_x = \vec{f}\cdot\hat{x}$$

Suppose we had a different basis, such as the $\hat{u}$ and $\hat{v}$ basis. The vector $\vec{f}$ is the same, but the components will be different (f_x and f_y are useless now, we want f_u and f_v):

$$\vec{f} = f_u\hat{u} + f_v\hat{v}$$

Once again we can find the components:

$$\hat{u}\cdot\vec{f} = f_u\hat{u}\cdot\hat{u} + f_v\hat{u}\cdot\hat{v} \to f_u = \hat{u}\cdot\vec{f}$$

We can write down a vector using the components in the basis:

$$\hat{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

But we don't need this, we can just use the facts that $\hat{x}\cdot\hat{x} = 1$, $\hat{x}\cdot\hat{y} = 0\hat{y}\cdot\hat{x}$, and $\hat{y}\cdot\hat{y} = 1$.

If we want to be able to compute the dot product of two arbitrary vectors $\vec{f}$ and $\vec{g}$:

$$\vec{f}\cdot\vec{g} = (f_x\hat{x} + f_y\hat{y})\cdot(g_x\hat{x} + g_y\hat{y}) = f_xg_x\hat{x}\cdot\hat{x} + f_xg_y\hat{x}\cdot\hat{y} + f_yg_x\hat{y}\cdot\hat{x} + f_yg_y\hat{y}\cdot\hat{y}$$

Now using the facts that we know, we have that

$$\vec{f} \cdot \vec{g} = f_x g_x + f_y g_y$$

Which is what we expected.

We can write this in a more slick notation, which generalizes to any number of basis vectors:

$$\vec{f} = \sum_m f_m \hat{m} \quad \vec{g} = \sum_n g_n \hat{n}$$

We can take the dot product:

$$\vec{f} \cdot \vec{g} = \sum_m \sum_n f_m g_n \hat{m} \cdot \hat{n}$$

We can now use the relationship between the basis vectors, when we know that $\hat{m} \cdot \hat{n} = \delta_{mn}$:

$$\vec{f} \cdot \vec{g} = \sum_{mn} f_m g_n \delta_{mn}$$

Also note that these bases are complete, which means that any 2D vector can be written in terms of the basis vectors. However, for 3D, $\hat{x}$ and $\hat{y}$ are not enough, we need a third orthogonal vector, $\hat{z}$. For higher dimensions, we run out of letters, so we switch to $\hat{e}_1, \hat{e}_2, \hat{e}_3, \ldots, \hat{e}_n$. However, these are not the only complete set of basis vectors, we have other ones such as the DFT basis. We call the generalized dot product the inner product when talking about higher dimensions.

4.2 Dirac Bra-Ket Notation

A column vector $\vec{v}$ is written as $|v\rangle$, called a "ket". We write row vectors $\vec{u}^\dagger$ as $\langle u|$, known as a "bra". These are named this way as a pun, and when we take the inner product:

$$\langle u, v \rangle$$

Is also known as the bracket of u and v. We can write a vector in terms of its components:

$$|v\rangle = v_x |x\rangle + v_y |y\rangle$$

And we can write row vectors:

$$\langle v| = \langle x| v_x^* + \langle y| v_y^*$$

If we now compute the self inner product:

$$\langle v|v\rangle = v_x^* v_x \langle x|x\rangle + v_x^* v_y \langle x|y\rangle + v_y^* v_x \langle y|x\rangle + v_y^* v_y \langle y|y\rangle$$

4.3 Inner Product

We can compute the inner product of two vectors $|f\rangle$ and $|g\rangle$ in both directions, and we find that

$$\langle f|g\rangle = (\langle g|f\rangle)^*$$

The inner product is also bilinear. Suppose $|g\rangle = |1\rangle + |2\rangle$, where $|1\rangle$ and $|2\rangle$ are two arbitrary vectors. The inner product

$$\langle f|g\rangle = \langle f|1\rangle + \langle f|2\rangle$$

Similarly, if $|g\rangle = \alpha |1\rangle$, then

$$\langle f|g\rangle = \alpha \langle f|1\rangle$$

The same properties hold for $|f\rangle$:

$$|f\rangle = |1\rangle + |2\rangle \to \langle f|G\rangle = \langle 1|g\rangle + \langle 2|g\rangle$$

And (this one is slightly more tricky)

$$|f\rangle = \alpha |1\rangle \to \langle f|g\rangle = \alpha^* \langle 1|g\rangle$$

This is because to go from $|f\rangle$ to $\langle f|$ we have to take a complex conjugate.

An inner product space is a space in which we have vectors and there exists a function that acts like the inner product. Take for example 3D vectors. The inner product in this case is what we normally think of as the dot product:

$$\sum_{n=1}^{3} f_n g_n$$

Let us take another example. If we have a D-dimensional Euclidean complex-valued space, the inner product is

$$\sum_{n=1}^{D} f_n^* g_n$$

What about an infinite dimensional space, such as a space with infinite components, that is complex valued. We can still rely on $\langle m|n\rangle = \delta_{mn}$, but we have to do an infinite sum:

$$\sum_n f_n^* g_n$$

Suppose the domain is all real numbers in the range $(-\pi, \pi)$, and for each of them, we have a complex-valued component. The analogous inner product for this is an integral instead of a sum:

$$\int_{-\pi}^{\pi} f^*(n) g(n)\, dn$$

Note that we could change the range, and it would only change the bounds of the integral.

Suppose we have a set of vectors $\{|I\rangle\}$ that forms an orthogonal basis. That is:

$$\langle I|J\rangle = N_I^2 \delta_{IJ}$$

for any two vectors $|I\rangle$ and $|J\rangle$ in the set. We can compute the self-inner product:

$$\langle I|I\rangle = N_I^2 \delta_{II} = N_I^2$$

If we generate a vector $|i\rangle = \frac{1}{N_I} |I\rangle$, and do something similar for all vectors in the set:

$$\langle i|j\rangle = \frac{1}{N_I^*} \frac{1}{N_J} \langle I|J\rangle = \frac{N_I^2}{|N_I|^2} \delta_{IJ}$$

Suppose we are now given a vector $|f\rangle$, and we need to find the components in a particular basis, such as the $|i\rangle$ basis:

$$|f\rangle = \sum_i c_i |i\rangle$$

To compute the components, we do it the same way we did it before:

$$\langle j|f\rangle = \langle j| \sum_i c_i |i\rangle\rangle = \sum_i c_i \langle j|i\rangle$$

Where we have leveraged linearity of the inner product and summation. We know that $\langle j|i\rangle = \delta_{ij}$:

$$\langle j|f\rangle = c_j$$

If we have a space of complex valued components of vectors with the range $(-\pi, \pi)$:

$$\langle f|g\rangle = \int_{-\pi}^{\pi} f^*(x)g(x)\,dx$$

$$|f\rangle = \int_{-\pi}^{\pi} f(x)\,|x\rangle\ dx$$

How can we get the component of $|f\rangle$ at some value of $x = x_0$ (in the range $(-\pi, \pi)$). To do this, we take the inner product:

$$\langle x_0|f\rangle = \int_{-\pi}^{\pi} f(x)\,\langle x_0|x\rangle\ dx$$

We want this to be equal to $f(x_0)$, so what does $\langle x_0|x\rangle$ have to equal, it must equal $\delta(x - x_0)$:

$$\langle x|y\rangle = \delta(x - y)$$

If we compute some inner product with two kets:

$$|f\rangle = \int f(x)\,|x\rangle\ dx \quad |g\rangle = \int g(x)\,|x\rangle$$

$$\langle f|g\rangle = \left(\int f^*(x)\,\langle x|\ dx\right)\left(\int g(y)\,\langle y|\ dy\right) = \iint f^*(x)g(y)\,\langle x|y\rangle\ dy\,dy$$

Using the orthogonality relationship:

$$= \int f^*(x)g(x)$$

If we write a ket via the complex form:

$$|f\rangle = \sum_k c_k\,|k\rangle \quad |k\rangle = \int e^{ikx}\,|x\rangle\ dx$$

And if we now compute some inner product:

$$\langle p|k\rangle = \left(\int e^{-ipx}\,\langle x|\ dx\right)\left(\int e^{iky}\,|y\rangle\ dy\right) = \iint e^{i(ky-px)}\,\langle x|y\rangle\ dx\,dy$$

$$= \int e^{i(k-p)x}\,dx = 2\pi\delta_{pk}$$

Thus we have that

$$\langle p|k\rangle = 2\pi\delta_{pk}$$

Then if we take some arbitrary inner product:

$$\langle k|f\rangle = 2\pi c_k$$

If we solve for c_k:

$$c_k = \frac{1}{2\pi}\langle k|f\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ikx} f(x)\,dx$$

This is just making clear that the Fourier expansion is just writing a vector out as a set of components in the Fourier Basis.

Let's take the inner product of two vectors in the Fourier Basis:

$$|f\rangle = \sum_k c_k\,|k\rangle \qquad |g\rangle = \sum_k d_k\,|k\rangle$$

Taking the inner product:

$$\langle f|g\rangle = \left(\sum_k c_k^* \langle k|\right)\left(\sum_p d_p\,|p\rangle\right) = \sum_{kp} c_k^* d_p \langle k|p\rangle$$

$$= 2\pi \sum_k c_k^* d_k$$

Thus we have that

$$\int f^*(x)g(x)\,dx = \langle f|g\rangle = 2\pi\sum_k c_k^* d_k$$

This gives us that

$$\sum_k c_k^* d_k = \frac{1}{2\pi}\int f^*(x)g(x)\,dx$$

This is the Plancherel Equation, which leads into Parseval's theorem, which we have seen before.

Let us look at the space where the range of x is $(-\infty,\infty)$. This should imply Fourier Transforms, rather than Fourier series. We have that

$$\langle f|g\rangle = \int_{-\infty}^{\infty} f^*(x)g(x)\,dx$$

And an arbitrary vector is written as

$$|f\rangle = \int f(x)\,|x\rangle\;dx$$

If we take the inner product of two of these arbitrary vectors:

$$\langle f|g\rangle = \iint f^*(x)g(x)\,\langle x|y\rangle\;dx\,dy$$

That inner product is $\delta(x-y)$:

$$\int_{-\infty}^{\infty} f^*(x)g(x)\,dx$$

In the Fourier basis:

$$|k\rangle = \int_{-\infty}^{\infty} e^{ikx}\,|x\rangle\;dx$$

Taking the inner product:

$$\langle p|k\rangle = \iint e^{-ipy+ikx}\,\langle x|y\rangle\;dx\,dy = \int e^{-i(k-p)x}\,dx = 2\pi\delta(p-k)$$

Let's try to compute the coefficients in the Fourier basis:

$$\langle x|f\rangle = f(x)$$

$$\langle p|f\rangle = \langle p| \int_{-\infty}^{\infty} F(k)\, |k\rangle \, dk\rangle = \int F(k)\, \langle p|k\rangle \, dk = 2\pi \int \delta(p-k) F(k)$$
$$= 2\pi F(p)$$

Thus

$$F(p) = \frac{1}{2\pi} \langle p|f\rangle = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ipx} f(x)\, dx$$

We see that this is exactly what we said for the Fourier transform.

In this Fourier basis, lets compute the inner product:

$$|f\rangle = \int F(k)\, |k\rangle \, dk$$

$$\langle f|g\rangle = \int F^*(k)\, \langle k| \, dk \int G(p)\, |p\rangle \, dp$$
$$= \iint F^*(k) G(p)\, \langle k|p\rangle = 2\pi \int F^*(k) G(k)\, dk$$

We can then set this equal to the other definition of writing the inner product:

$$\langle f|g\rangle = \int f^*(x) g(x)\, dx$$

This once again gives us Plancherel's Theorem/Equation.

If we have a space of functions with domain $(-1, 1)$ that are complex valued:

$$\langle f|g\rangle = \int_{-1}^{1} f^*(x) g(x)\, dx$$

$$|f\rangle = \sum_l a_l\, |l\rangle$$

Where

$$|l\rangle = \int_{-1}^{1} P_l(x)\, |x\rangle \, dx$$

And

$$\langle l|m\rangle = \frac{2}{2l+1} \delta_{lm}$$

Computing the inner product of two arbitrary functions:

$$\langle j|f\rangle = \sum_l a_l\, \langle j|l\rangle = \sum_l a_l \frac{2}{2l+1} \delta_{jl} = \frac{2a_j}{2j+1}$$

Using the other definition:

$$\frac{2a_j}{2j+1} = \int_{-1}^{1} P_j^*(x) f(x)\, dx$$

So when should we use the Fourier Basis and when should we use P_l. They're both complete, and orthogonal, and both decompose a function into sets of components. How do we choose which one to use?

We know that $\langle u|v\rangle$ is an inner product, but what is $|v\rangle\langle u|$? If we pattern match using linear algebra, if we have

$$(|v\rangle\langle u|)\,|w\rangle$$

This is similar to a matrix acting on a vector in linear algebra. We know that given an orthonormal basis $\{|i\rangle\}$, for any two basis vectors, $\langle i|j\rangle = \delta_{ij}$. For any arbitrary matrix M, the matrix element M_{ij} is related to $|i\rangle$ and $|j\rangle$. To get the value of M_{ij}, we can do

$$\langle i|M|j\rangle$$

We can write the matrix generally:

$$M = \sum_{ij} |i\rangle\, M_{ij}\, |j\rangle$$

We can apply this matrix against a ket:

$$M\,|17\rangle = \sum_{ij} |i\rangle\, M_{ij}\, \langle j|17\rangle = \sum_{ij} |i\rangle\, M_{ij}\delta_{j,17}$$
$$= \sum_{i} |i\rangle\, M_{i,17}$$

What about the claim of completeness for these bases? A basis $\{|i\rangle\}$ is complete if every vector can be written

$$|v\rangle = \sum_i c_i\,|i\rangle$$

We have shown that

$$\langle j|v\rangle = \langle j|\sum_i c_i|i\rangle = \sum_i c_i\,\langle j|i\rangle = \sum_i c_i\delta_{ij} = c_j$$

Thus we have that $c_j = \langle j|v\rangle$, and we can then write out $|v\rangle$:

$$|v\rangle = \sum_i \langle i|v\rangle\,|i\rangle = \sum_i |i\rangle\,\langle i|v\rangle = \left(\sum_i |i\rangle\,\langle i|\right)|v\rangle$$

This tells us that

$$|v\rangle = M\,|v\rangle$$

If this is true for all $|v\rangle$, then M must be the identity, and thus

$$I = \sum_i |i\rangle\,\langle i|$$

We can look at the derivative operator:

$$D = \int_{-\infty}^{\infty} |x\rangle\,\frac{d}{dx}\,\langle x|\;dx$$

Lets act this on the vector f:

$$D\,|f\rangle = \int dx\;|x\rangle\,\frac{d}{dx}\,\langle x|\int dy\,f(y)\,|y\rangle = \int dx\;|x\rangle\,\frac{d}{dx}\int dy\;\langle x|y\rangle$$
$$= \int dx\;|x\rangle\,\frac{d}{dx}\int dy\,f(y)\delta(x-y) = \int dx\;|x\rangle\,\frac{d}{dx}f(x)$$

4.4 Eigenvalue Problem

If we are given a matrix M, the goal is to find vectors and corresponding numbers such that

$$M\vec{v} = \lambda\vec{v}$$

$\vec{v}$ is an eigenvector and λ is an eigenvalue. The set of $\vec{v}$s forms a complete basis:

$$\vec{u} = \sum_n c_n \vec{v}_n$$

We can write this with kets:

$$M\left|v\right\rangle = \lambda\left|v\right\rangle$$

And we have that

$$\langle x|M|v\rangle = \lambda\left\langle x|v\right\rangle$$

5 Partial Differential Equations

ODEs have 1 independent variable, while PDEs have more than 1. The order of the equation is the highest number of derivatives in any term. For example, the equation

$$\partial_x^2 T = \frac{1}{\alpha^2}\partial_t T$$

is a second order PDE.

An equation that is linear can be written in terms of an operator acting on a function:

$$Ly = f$$

A solution to a PDE is any function that when plugged in as the dependent variable gives an equality. Remember that the independent variables are inputs, things we take for granted, such as x, y, z, and t. Dependent variables are output, properties/behavior of the system of interest, such as temperature, pressure, deformation, or other possible quantities. These are all fields. Fields are physical quantities that have values for every combination of independent variables that we're interested in. For example, we could have a rod, whose temperature would be some function $T(x,t)$. If we specify the position and the time, we can always assign a value for the temperature at that time and place.

We have other types of fields, such as force fields:

$$\vec{F}(\vec{r},t)$$

Where the force is dependent on the position and time. Note that the force is a vector, so this is a vector field. The temperature case is a scalar field, where the quantity returned by the field is just a scalar. Pressure is another example of a scalar field, where the pressure at a given point and time is just a single number. Other examples of vector fields are EM fields, like the electric field or the magnetic field.

There are other fields too, such as spinors (used in QFT) and tensors (used in GR).

How do we solve these PDEs? One method is to have experience, and just guess solutions via experience. Another possible method is to ask a computer. Then there are special techniques for linear PDEs.

Linear ODEs were good because they obeyed the superposition rule, where if we found two solutions, we could just add them together to get a solution. For PDEs, we will find a large number of solutions that are linearly independent (orthogonal). If we can find a good basis, then we can express every solution in terms of that basis.

Let's see some examples.

Laplace's equation:

$$\nabla^2 u = 0$$

Helmholtz's Equation:

$$\nabla^2 u + k^2 u = 0$$

The Wave equation:

$$\nabla^2 u = \frac{1}{v^2}\partial_t^2 u$$

The Diffusion/Heat equation:

$$\nabla^2 u = \frac{1}{\alpha^2}\partial_t u$$

The Schrodinger equation:

$$\nabla^2 \psi + V\psi = i\partial_t \psi$$

Note that all of these equations can be rewritten as $Lu = 0$. These are all homogeneous equations. We also have Poisson's equation, which is inhomogeneous:

$$\nabla^2 u = f(x, y, z)$$

Imagine a string vibrating in 1 direction as a function of time:

$$\partial_x^2 u = \frac{1}{v^2}\partial_t^2 u$$

u is a function of both x and t. Let us separate variables to solve this wave equation. We say that

$$u(x, t) = X(x)T(t)$$

Plugging this into the wave equation:

$$T(t)\partial_x^2 X(x) = \frac{1}{v^2}X(x)\partial_t^2 T(t)$$

Now we divide by XT:

$$\frac{1}{X}\partial_X^2 X = \frac{1}{v^2}\frac{1}{T}\partial_t^2 T$$

The left side is a function of x only. The right side is a function of t only. Both sides of this must be the same constant:

$$\frac{1}{X}\partial_x^2 X = \lambda = \frac{1}{v^2 T}\partial_t^2 T$$

We can now solve the first equation:

$$\partial_x^2 X = \lambda X$$

And solve the second equation:

$$\partial_t^2 T = v^2 \lambda T$$

From these, and the fact that we want oscillatory solutions, we get that

$$X = c_+ e^{ikx} + c_- e^{-ikx} \quad T = d_+ e^{ikvt} + de^{-ikvt}$$

We now need to multiply the solutions together to get u:

$$X_k T_k = c_+ d_+ e^{ik(x+vt)} + c_+ d_- e^{ik(x-vt)} + c_- d_+ e^{-ikx(x-vt)} + c_- d_- e^{-ik(x+vt)}$$

We see that we can group these into terms with $x - vt$ and those with $x + vt$, which gives us the right and left moving waves, respectively. Which values of k are allowed? We can use any value of k that we want, whether it be negative, positive, or 0. Thus we have the general solution of the wave equation, summing over all possible allowed values of k:

$$u = \int_{-\infty}^{\infty} dk \, \left[L(k) e^{ik(x+vt)} + R(k) e^{ik(x-vt)} \right]$$

We haven't left out the negative terms because we're integrating from $-\infty$.

Does this really solve the wave equation? One way to check is to take the derivatives:

$$\partial_x^2 u = \int dk \, (ik)^2 \left[L(k) e^{ik(x+vt)} + R(k) e^{ik(x-vt)} \right]$$

$$\partial_t^2 u = \int dk \, (ikv)^2 \left[L(k) e^{ik(x+vt)} + R(k) e^{ik(x-vt)} \right] = v^2 \partial_x^2 u$$

In terms of initial conditions, we need to specify $u(x, 0)$, and $\partial_t u(x, 0)$.

Why did we pick $\lambda = -k^2$ in the second equation? This is because we can't have exponentially growing functions, due to physical limitations of the situation. This gives us the boundary conditions:

$$u(x = \pm\infty) \neq \infty \quad u(t = \pm\infty) \neq \infty$$

There are other boundary conditions as well. The Dirichlet Boundary Conditions (function equal to certain value at certain place):

$$u(a) = c$$

The Neumann Boundary Condition (specify derivative at a particular place):

$$u'(a) = 0$$

Periodic Boundary Conditions (going some period away will loop the function):

$$u(x + 2\pi) = u(x)$$

Consider a 2π-periodic String. We have the boundary condition $u(x, t) = u(x + 2\pi, t)$:

$$X(x + 2\pi) = X(x)$$

Thus we have that $e^{ik(x+2\pi)} = e^{ikx}$, which gets us that $e^{2\pi ik} = 1$. Ths tells us that k must be an integer. When we go to construct the generic function, we can't use an integral, and instead must use an infinite sum instead:

$$u = \sum_{k=-\infty}^{\infty} l_k e^{ik(x+vt)} + r_k e^{ik(x-vt)}$$

What if we clamp down the string at $x = 0$ and $x = \pi$. Instead of using e^{ikx}, we can use sin and cos instead (can't use the exponential because its never 0):

$$u = \sum_{k=-\infty}^{\infty} b_k \sin(kx) e^{ikvt}$$

Where the sin is from the position ODE and the exponential is from the time ODE. We don't have another term because we can split this sum:

$$u(x,t) = \sum_{k=1}^{\infty} \sin(kx) b_k e^{ikvt} + \sum_{k=1}^{\infty} \sin(-kx) b_{-k} e^{-ikvt}$$

Now using the fact that sine is odd, and combining the two sums:

$$= \sum_{k=1}^{\infty} \sin(kx) \left(b_k e^{ikvt} - b_{-k} e^{-ikvt} \right)$$

And we see that we have the expected solution.

Let's take the example of a pipe open at $x = 0$ and $x = \pi$:

$$\partial_x^2 p = \frac{1}{v^2} \partial_t^2 p$$

We can solve this with the same construction (4 functions, the plus/minuses are independent of each other):

$$p = e^{\pm ikx} e^{\pm ikvt}$$

We know that $\partial_x p(0,t) = 0 = \partial_x p(\pi, t)$ so we need a solution that obeys those. Taking the x derivative of p:

$$\partial_x p \propto \pm ik e^{\pm ikx}$$

This is never 0, but we know that $\partial_x(\cos(kx)) = 0$ when $x = 0$, so this implies that $\cos(kx)$ is the spatial part of the solution:

$$p \sim \cos(kx) e^{\pm ikvt}$$

Looking at the second boundary condition:

$$\partial_x p(\pi, t) = -k \sin(k\pi) e^{\pm ikvt} = 0$$

Thus we need $k \in \mathbb{Z}$. The choices for k are called the spectrum. This gives the solution:

$$p = \frac{a_0}{2} + \sum \cos(kx)(a_k e^{ikvt} + a_{-k} e^{-ikvt})$$

In general, if we have a PDE that is linear and homogeneous ($Lu = 0$), we can find a basis via separation of variables, giving us separation constants. We can then apply the boundary conditions, which takes the basis and removes certain basis vectors from our combination. The solution is just the sum over the allowed separation constants of the basis functions for the separation constants, with some arbitrary coefficients.

5.1 Laplace's Equation

Laplace's equation is

$$\nabla^2 u = 0$$

Where ∇^2, known as the Laplacian is defined as

$$\nabla^2 = \partial_x^2 + \partial_y^2 + \partial_z^2$$

in 3 dimensions. In n dimensions:

$$\nabla^2 = \sum_{i=1}^{n} \partial_{x_i}^2$$

In two dimensions, we can use separation of variables:

$$u = X(x)Y(y)$$

Now inserting this into the equation:

$$(\partial_x^2 + \partial_y^2)XY = Y\partial_x^2 X + X\partial_y^2 Y = 0$$

$$\frac{\partial_x^2 X}{X} + \frac{\partial_y^2 Y}{Y} = 0$$

The first piece only depends on x, and the second only depends on y, and thus they must be constants:

$$\partial_x^2 X = \lambda X \quad \partial_y^2 Y = -\lambda Y$$

If we want periodic solutions in X, we would require exponential growth in Y, and vice versa.

What is Laplace's equation? Recall that

$$\partial_x^2 f = \lim_{h\to 0} \frac{f(x+h) - 2f(x) + f(x-h)}{h^2}$$

If this is 0, then

$$f(x+h) - 2f(x) + f(x-h) = 0$$

And thus

$$f(x) = \frac{f(x+h) + f(x-h)}{2}$$

We see that values at x equal the average of the surrounding values. If we worked it out for two dimensions:

$$\partial_x^2 f + \partial_y^2 f = 0 \to f(x,y) = \frac{f(x+h,y) + f(x-h,y) + f(x,y+h) + f(x,y-h)}{4}$$

We see that the intuition of an average still holds. The functions that satisfy this equation are functions that can be made via deforming a rubber sheet. They need to be smooth and elastic.

Helmholtz's equation is given by

$$\nabla^2 u + k^2 u = 0$$

One can think of Laplace's equation as a special case of this equation. This shows up when studying the wave equation:

$$\nabla^2 u - \frac{1}{v^2}\partial_t^2 u$$

$$\nabla^2 u - \frac{1}{\alpha^2}\partial_t u$$

If we say that $u = X(x, y, z)T(t)$, we find that $T(t)$ is an exponential, and we will find that we cna separate out the time dependent portion.

Picking god coordinate systems is also an issue. Suppose we have a cylindrical system. In this case, the boundary conditions are best expressed in cylindrical coordinates. In cylindrical, we use r, ϕ, and z:

$$r^2 = x^2 + y^2 \quad \tan\phi = \frac{y}{x} \quad z = z$$

The Laplacian in cylindrical coordinates is not equal to the natural expectation:

$$\nabla^2 \neq \partial_r^2 + \partial_\phi^2 + \partial_z^2$$

The z partial does match, so that's not an issue, but the other two need to match up. Using the chain rule:

$$\partial_x^2 f = \partial_x(\partial_r f \partial_x r + \partial_\phi \partial_x \phi)$$

Now using the product rule and the chain rule:

$$= (\partial_r^2 f(\partial_x r)^2 + \partial_r f \partial_x^2 r + \partial_\phi^2 f(\partial_x \phi)^2 + \partial_\phi f \partial_x^2 \phi)$$

We could then use the transformations. If we do it all out, we find that

$$\nabla^2 u = \left(\frac{1}{r}D_r r D_r + \frac{1}{r^2}D_\phi^2 + D^2 z\right) u$$

If we do separation of variables on

$$-k^2 u = \nabla^2 u$$

in cylindrical, we eventually get

$$\lambda_z Z = \partial_z^2 Z$$
$$\lambda_\phi \phi = \partial_\phi^2 \phi$$
$$-\lambda_r R = \partial_r^2 R + \frac{1}{r}\partial_r R + \frac{\lambda_\phi}{r^2} R$$
$$\lambda_r = \lambda_z + k^2$$

Also note that $u(\phi) = u(\phi + 2\pi)$ (since it wraps around), and we say that $\lambda_\phi = -m^2$. The solutions to these conditions are known as Bessel functions. These functions lead to Hankel functions.