Contents

1 Introduction

This course is about the physics of vibrations and waves. We don't learn any new laws of physics, we simply build off of what we already know. The waves we talk about are related to mechanical waves and electromagnetic waves. We develop methods of applying Newton's Laws and Maxwell's equations to oscillations. We then talk about how vibrations being transmitted through space create waves.

We start with mechanical oscillation, including springs, pendulums, and coupled oscillators. We also talk about electromagnetic oscillators, such as inductor and capacitor circuits (LC circuits) and circuits including resistors (RLC circuits). We then transition to mechanical waves, such as sound waves. We then transition to EM waves, like waves in cables, waves in space, and optics (interference, diffraction).

In this course, we will also learn to apply mathematical methods such as methods for solving linear diffeqs, complex algebra, Taylor expansions, exponential functions (e^x), and Fourier series.

2 Simple Harmonic Motion

2.1 Equation of Motion

The simplest oscillator is a mass and a spring. We have a mass m and a spring with some spring constant s (sometimes also shown as k). We assume that there is no friction, and we displace the mass from its equilibrium state some distance x. The spring wants to restore the system back to equilibrium, so it exerts a restoring force:

$$F = -sx$$

From here, we use Newton's Law, $F = ma$:

$$m\frac{d^2x}{dt^2} = -sx$$

Rewriting, we get the core equation of all SHM:

$$\boxed{m\frac{d^2x}{dt^2} + sx = 0}$$

This is a linear differential equation. This is linear because there is no higher order of x. We use this property as we move along. This equation is known as the equation of motion. Our job is now to solve this to find x as a function of time. Note that when we say that the restoring force is linear, this only occurs for small displacements, as the spring will behave differently if we for example, pull the spring out very far. We are also ignoring the mass of the spring, which is impossible in the real world.

2.2 Solution

To solve this equation, we can guess a solution, $x = A\cos\omega t$. If we do this:

$$\frac{dx}{dt} = -A\omega\sin\omega t$$

$$\frac{d^2x}{dt} = -A\omega^2\cos\omega t$$

Plugging these in, we get that

$$-Am\omega^2 \cos \omega t + sA \cos \omega t = 0$$

This gives us that

$$\boxed{\omega = \sqrt{\frac{s}{m}}}$$

ω is known as the **angular frequency**, and is completely independent of the amplitude of the oscillation (based on x). Our solution is a periodic function:

$$x = A \cos \omega t = A \cos(\omega t + \omega T)$$

If $\omega T = 2\pi$, we see that we are back to where we started. $T = \frac{2\pi}{\omega}$ is the period of oscillations. Via dimensional analysis, we know that T is in units of time, and ω has units of inverse seconds. Rewriting in terms of ω:

$$\omega = \frac{2\pi}{T} = 2\pi \frac{1}{T} = 2\pi\nu$$

Where $\nu = \frac{1}{T}$, and is known as the **frequency** (Note that this is different from angular frequency). This is the number of times the system oscillates per second. Angular frequency is just 2π times this.

We also notice that $x = B \sin \omega t$ is also a solution to the diffeq. This means that we can construct a general solution, a linear combination of the two solutions that we have:

$$x = A \cos \omega t + B \sin \omega t$$

Where A and B are any arbitrary constants. These constants can be determined when we take into account the initial conditions of the system, such as initial position and velocity (x_0 and $\dot{x}_0$). For example, if we know that at $t = 0$, $x = 0$. We can plug in $t = 0$ into $x(t)$:

$$x(0) = A + B(0) = 0$$

This tells us that in this case, $A = 0$, so our equation will be

$$x(t) = B \sin \omega t$$

To find what B is, we take the derivative to get $\dot{x}(t)$:

$$\dot{x}(t) = B\omega \cos \omega t$$

Plugging in the fact that at $t = 0$, $\dot{x} = v_0$:

$$B = \frac{v_0}{\omega}$$

This would give us the final equation for this specific case:

$$x = \frac{v_0}{\omega} \sin \omega t$$

We can also replace A and B with something else:

$$A = a \sin \phi \qquad B = a \cos \phi$$

Note that we can define them to be whatever we want. Plugging these new values in:

$$x = a\sin\phi\cos\omega t + a\cos\phi\sin\omega t$$

$$= a(\sin\phi\cos\omega t + \cos\phi\sin\omega t)$$

Using angle addition identities:

$$x = a\sin(\omega t + \phi)$$

Note that this depends on the arbitrary choice of how we represent A and B, but it gives us an equation with the same quantities, and only one trig function, with an additional term ϕ, known as the **phase shift**. $(\omega t + \phi)$ is known as the **phase** of the oscillation. a is known as the **amplitude** of the oscillation.

Note that the phase is dependent on where the system starts off, as the phase offsets the sinusoid, which we can see if we look at $t = 0$:

$$x(0) = a\sin\phi$$

We see that ϕ determines where the function reaches its maxima, and tyhe function has the same amplitude a.

Lets look at what the velocity, $\dot{x}(t)$ looks like, and what the acceleration $\ddot{x}(t)$ looks like:

$$\dot{x} = a\omega\cos(\omega t + \phi)$$

$$\ddot{x} = -a\omega^2\sin(\omega t + \phi)$$

We see that $\dot{x}$ is 90 degrees out of phase with x, due to the difference between sine and cosine, and we can see that the acceleration is a whole 180 degrees out of phase, due to the negative sign.

2.3 Energy in the System

Let's now look at the energy in this system. We have two different types of energy in the system, potential and kinetic. To calulate the potential, we look at the force we have, $F = -sx$. When we displace the spring, we are doing work against the restoring force, which is the potential energy:

$$PE = -\int_0^x -sx = \frac{1}{2}sx^2$$

To find the kinetic energy, we just use the regular equation:

$$KE = \frac{1}{2}m\dot{x}^2$$

We know that $\dot{x} = a\omega\cos(\omega t + \phi)$, so we can plug it in. This leaves us with an expression for potential energy:

$$PE = \frac{1}{2}sa^2\sin^2(\omega t + \phi)$$

Using the fact that $\omega^2 = \frac{s}{m}$:

$$PE = \frac{1}{2}m\omega^2 a^2\sin^2(\omega t + \phi)$$

and an expression for kinetic energy:

$$KE = \frac{1}{2}m\omega^2 a^2\cos^2(\omega t + \phi)$$

Notice that both of them are oscillating, but if we look at the sum of the expressions, the sum of the total energy in the system, we see that they sum to a constant:

$$KE + PE = \frac{1}{2}m\omega^2 a^2$$

This tells us exactly what we wanted, that the energy in the system is conserved.

2.4 Potential Functions

We just showed that PE, sometimes also written as V, is equal to $\frac{1}{2}m\omega^2 x^2$. We know that $F = -\frac{dV}{dx}$, which is just $-sx$, or $-m\omega^2 x$. Taking the second derivative:

$$\frac{d^2V}{dx^2} = m\omega^2$$

This tells us that $\omega^2 = \frac{1}{m}\frac{d^2V}{dx^2}$. Why do we care about this?

What if we had a problem where we knew the potential, but not the force? If we plotted the potential against x and the graph was periodic, and we placed a mass in a trough, the mass would oscillate back and forth. If we have some $V(x)$, we have some areas where there are local equilibriums, such as the bottom of a trough. We look at the Taylor expansion of $V(x)$:

$$V(x) = V(x_0) + \dot{V}(x_0)(x - x_0) + \frac{\ddot{V}(x_0)(x - x_0)^2}{2!} + \dots$$

At equilibrium (a maxima), the value of $\dot{V}(x_0)$ will be 0. This means that we can approximate:

$$V(x) - V(x_0) \approx \frac{\ddot{V}(x_0)(x - x_0)^2}{2}$$

This means that near all minima, we can have an oscillation that is related to the second derivative:

$$\boxed{\omega^2 = \frac{1}{m}\ddot{V}(x_0)}$$

3 Damped Oscillations

Damped oscillations have an additional force, which isn't linear, and that opposes the motion:

$$F_D = -r\dot{x}$$

where r is a constant, similar to s. Using $F = ma$, we can get the diffeq for the situation:

$$m\ddot{x} = -r\dot{x} - sx$$

$$\boxed{m\ddot{x} + r\dot{x} + sx = 0}$$

This is more complicated than SHM, but is still a linear diffeq. The simple example of this is a mass on a spring with friction. Note that in most physical cases, it is proportional to v, but in some cases you could see proportionality to v^2. The task is to find a solution to this diffeq. Looking at circuits, this is a LRC circuit:

$$L\frac{dI}{dt} + IR + \frac{q}{c} = 0$$

$$\boxed{L\ddot{q} + R\dot{q} + \frac{q}{c} = 0}$$

3.1 Extension of Math Toolkit (Complex Numbers)

3.1.1 Basics of Complex Numbers

We start off by introducing complex variables, which starts with the definition of i:

$$i = \sqrt{-1}$$

We can construct a complex number:

$$z = a + bi$$

We can do algebra and geometry with these. For example, each real number represents a point on an axis, whereas a complex number can be represent a dot on a plane, or a vector from the origin to the point. If we take the magnitude of that vector, we have a definition of the magnitude of a complex number:

$$|z| = \sqrt{a^2 + b^2}$$

Using this, we can rewrite the complex numbers by using the angle between the origin and the vector:

$$z = |z|(\cos\theta + i\sin\theta)$$

We also have an expression for the tangent of the angle:

$$\tan\theta = \frac{b}{a}$$

Let's do some algebra with these complex numbers. Take the following complex numbers:

$$z_1 = a_1 + b_1 i \qquad z_2 = a_2 + b_2 i$$

$$z_1 + z_2 = (a_1 + a_2) + i(b_1 + b_2)$$

The same logic holds for subtraction. For multiplication, we just FOIL:

$$z_1 z_2 = a_1 a_2 + i a_1 b_2 + i a_2 b_1 + i^2 b_1 b_2 = a_1 a_2 - b_1 b_2 + i(a_1 b2 + a_2 b_1)$$

For division:

$$\frac{z_1}{z_2} = \frac{z_1 + ib_1}{a_2 + ib_2}$$

To do this, we have to define the complex conjugate:

$$z^* = a - ib$$

Note that

$$zz^* = a^2 + b^2$$

so $|z| = \sqrt{zz^*}$. Moving back to division, multiplying both the top and bottom by the complex conjugate:

$$\frac{(a_1 + ib_1)(a_2 - ib_2)}{(a_2 + ib_2)(a_2 - ib_2)}$$

$$= \frac{(a_1 + ib_1)(a_2 - ib_2)}{a_2^2 + b_2^2}$$

Another feature that is nice to know:

$$z = a + ib \qquad z^* = a - ib$$

$$Re(z) = a = \frac{z + z^*}{z}$$

$$Im(z) = b = \frac{z - z^*}{zi}$$

3.1.2 Natural Exponential Function

The natural exponential function is e^x. Remember that the definition of e is

$$e = \sum_{n=0}^{\infty} \frac{1}{n!}$$

And we can Taylor expand e^x:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

The most beautiful part of e^x is the property that the derivative of e^x is itself:

$$\frac{de^x}{dx} = e^x$$

3.1.3 Complex Exponential Function

$$e^{ix} = 1 + ix + \frac{i^2x^2}{2!} + \dots$$

Separating the even and odd terms:

$$e^{ix} = (1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \dots) + i(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots)$$

We see that this is two infinite series that look familiar, the first being the Taylor expansion for cos, and the second being the Taylor expansion for sin:

$$\boxed{e^{ix} = \cos x + i \sin x}$$

Feynman called this the most remarkable formula in math. Looking at e^{-ix}:

$$e^{-ix} = \cos x - i \sin x$$

From these, we can write definitions of the trig functions:

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}$$

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i}$$

We can then have a little bit of fun:

$$e^{i\pi} = \cos \pi + i \sin \pi = -1$$

Rewriting the left side:

$$(e^{i\pi/2})^2 = -1$$

Taking the square root:

$$e^{i\pi/2} = \sqrt{-1} = i$$

This tells us that:

$$\sqrt{i} = e^{i\pi/4} = \cos(\pi/4) + i \sin(\pi/4)$$

$$= \frac{1+i}{\sqrt{2}}$$

Note that we can also rewrite any arbitrary complex number:

$$z = a + ib = |z|(\cos\theta + i\sin\theta) = |z|e^{i\theta}$$

$$\boxed{z = |z|e^{i\theta}}$$

We can now redefine the multiplication of complex numbers:

$$z_1 = |z_1|e^{i\theta_1} \qquad z_2 = |z_2|e^{i\theta_2}$$

$$z_1 z_2 = |z_1||z_2|e^{i(\theta_1+\theta_2)}$$

Likewise, division:

$$\frac{z_1}{z_2} = \frac{|z_1|}{|z_2|}e^{i(\theta_1-\theta_2)}$$

Let's look at the logarithm of an arbitrary complex number:

$$\ln z = \ln(|z|e^{i\theta}) = \ln|z| + i(\theta)$$

Note that this isn't completely correct, because if we add 2π to θ, the result is the same, so the correct definition is:

$$\ln z = \ln|z| + i(\theta + 2n\pi)$$

3.1.4 Roots of Unity

Lets look at the cube roots of 1:

$$1^{\frac{1}{3}} = (e^{i2n\pi})^{\frac{1}{3}} = e^{i\frac{2n\pi}{3}}$$

3.1.5 SHM Using Complex Numbers

Let's move back to the SHM equation:

$$m\ddot{x} + sx = 0$$

We can claim that a possible solution is

$$x = Ae^{i\omega t}$$

Plugging this in:

$$\dot{x} = Ai\omega e^{i\omega t}$$

$$\ddot{x} = Ai^2\omega^2 e^{i\omega t} = -A\omega^2 e^{i\omega t}$$

$$-mA\omega^2 e^{i\omega t} + Ase^{i\omega t} = 0$$

$$-m\omega^2 + s = 0$$

$$\omega^2 = \frac{s}{m}$$

We can also look at $x = Be^{-i\omega t}$, and we see that this is also a solution, giving us a general solution:

$$x = Ae^{i\omega t} + Be^{-i\omega t}$$

Note that x has to be real, so to get the real solution, we can take the real portion of x, or we can manipulate the solution to become a real number, by choosing values for A and B:

$$A = B^* = \frac{C}{2i}e^{i\phi}$$

Plugging these in:

$$X = \frac{C}{2i}e^{i\phi}e^{i\omega t} - \frac{C}{2i}e^{-i\phi}e^{-i\omega t}$$

Doing out the math, we eventually get

$$x = C\sin(\omega t + \phi)$$

Note that we can get the different possible solutions by choosing different values for A and B.

3.1.6 Beats

Let's say we have a system that is oscillating under 2 different vibrations:

$$x_1 = A_1e^{i(\omega t+\phi_1)} \qquad x_2 = A_2e^{i(\omega t+\phi_2)}$$

$$x = x_1 + x_2 = A_1e^{i(\omega t+\phi_1)} + A_2e^{i(\omega t+\phi_2)}$$
$$= e^{i(\omega t+\phi_1)}(A_1 + A_2e^{i(\phi_2-\phi_1)})$$

We can see that the things in the parentheses is the amplitude of the oscillation. lets look at another case:

$$x_1 = Ae^{i\omega_1 t} \qquad x_2 = Ae^{i\omega_2 t}$$
$$x = A(e^{i\omega_1 t} + e^{i\omega_2 t})$$

We can rewrite ω_1:

$$\omega_1 = \frac{(\omega_1+\omega_2)}{2} + \frac{\omega_1-\omega_2}{2}$$

and also rewrite ω_2:

$$\omega_2 = \frac{\omega_1+\omega_2}{2} - \frac{\omega_1-\omega_2}{2}$$

We can then rewrite x:

$$x = Ae^{\frac{i(\omega_1+\omega_2)t}{2}}(e^{i(\omega_1-\omega_2)t} + e^{-i(\omega_1-\omega 2)t})$$

This is just:

$$x = 2Ae^{i(\omega_1+\omega_2)t}\cos(\omega_1-\omega_2)t$$

Taking just the real part of this:

$$x = 2A\cos(\frac{(\omega_1+\omega_2)t}{2})\cos\frac{(\omega_1-\omega_2)t}{2}$$

This is what we call beats, where the system is oscillating with two frequencies, where we see a rapid oscillation enveloped with a slow oscillation.

3.2 Solving Damped Oscillation

Moving back to the damped oscillation, we can try a solution:

$$x = ce^{\alpha t}$$

Which we can plug into the equation:

$$\dot{x} = c\alpha e^{\alpha t} \quad \ddot{x} = c\alpha^2 e^{\alpha t}$$

$$mc\alpha^2 e^{\alpha t} + cr\alpha e^{\alpha t} + sce^{\alpha t} = 0$$

$$m\alpha^2 + r\alpha + s = 0$$

This is a quadratic, which we can solve:

$$\alpha = \frac{-r}{2m} \pm \sqrt{\frac{r^2}{4m^2} - \frac{s}{m}}$$

3.3 Underdamped Oscillation

Notice that we see $\frac{s}{m}$ is here (ω_0^2). Let's look at some interesting cases. One case (**Underdamped Oscillation**) is when

$$\frac{r^2}{4m^2} < \frac{s}{m}$$

Then we see that the expression underneath the square root is negative, so we will have a complex solution:

$$\sqrt{\frac{r^2}{4m^2} - \frac{s}{m}} = (\omega_0^2 - \frac{r^2}{4m^2})^{\frac{1}{2}} i$$

We call this value on the right ω:

$$\omega = (\omega_0^2 - \frac{r^2}{4m^2})^{\frac{1}{2}}$$

The two values of α will be

$$\alpha_1 = \frac{-1}{2m} + i\omega \qquad \alpha_2 = \frac{-1}{2m} - i\omega$$

Plugging these values back into x:

$$x_1 = c_1 e^{\frac{-r}{2m}t} e^{i\omega t}$$

$$x_2 = c_2 e^{\frac{-r}{2m}t} e^{-i\omega t}$$

We can see that ω is the angular frequency of a damped oscillation, and is different from ω_0. If we take the case that there is no damping, we see that $\omega = \omega_0$. Our general solution is

$$x = x_1 + x_2 = \frac{-r}{2m} t (c_1 e^{i\omega t} + c_2 e^{-i\omega t})$$

We can then say that $c_1 = c_2^* = \frac{c}{2i} e^{i\phi}$ (like what we did previously), we are left with

$$\boxed{x = ce^{-\frac{r}{2m}t} \sin(\omega t + \phi)}$$

As time passes in this damped oscillation, we see that the exponential term gets smaller over time, so the oscillation runs down (as we would expect when talking about a real world pendulum). Thinking about the energy in the system, we have some amount of energy in the system in the beginning, and as the oscillation decays, the energy also decays based on $E = \frac{1}{2}kx^2 + \frac{1}{2}m\dot{x}^2$. We see that energy will always be proportional to the amplitude squared, so as the amplitude decays:

$$A = Ce^{\frac{-r}{2m}t}$$

$$E = E_0 e^{\frac{-r}{m}t}$$

3.4 Oscillation Quantities

We have a quantity called the 'relaxation time', which is the amount of time it takes for the energy to drop by $\frac{1}{e}$:

$$\tau = \frac{m}{r}$$

Another quantity we have is the 'quality factor', the change in the phase of the oscillation for $\Delta t = \tau$:

$$Q = \frac{\omega m}{r}$$

We had that

$$\omega^2 = \omega_0^2 - \frac{r^2}{4m^2}$$

In the case that

$$\frac{r^2}{4m^2} \ll \omega_0^2$$

$$\omega \approx \omega_0$$

$$Q = \frac{\omega m}{r} \approx \frac{\omega_0 m}{r}$$

$$x = ce^{-\frac{r}{2m}t}\sin(\omega t + \phi) \approx ce^{\frac{-\omega_0 t}{wQ}}\sin(\omega t + \phi)$$

$$\boxed{E = -c^2 e^{\frac{-\omega_0 t}{Q}}}$$

We see that Q tells us that if Q is large, damping is very small, and the energy loss is also small. If $\frac{\omega_0 t}{Q} = 1$, Q measures the change in the phase for the simple oscillation.

If we rewrite the equations of motion:

$$\ddot{x} + \frac{\omega_0}{Q}\dot{x} + \omega_0^2 x = 0$$

This is interesting in cases where we have very large Q, where we can ignore the second term entirely. We can also show that

$$\frac{Q}{s\pi} = \frac{E}{-\Delta E}$$

The right side is the energy loss in a period of oscillation (Sec. 2.4 P&R).

3.5 Overdamped Systems

Moving back to calculating α, we have the case where

$$\frac{r^2}{4m^2} > \omega_0^2$$

This is **Overdamping**. We see that both α_1 and α_2 are real, so we can write the solution out:

$$x = c_1 e^{\alpha_1 t} + c_2 e^{\alpha_2 t}$$

We have no oscillation occurring. Let's analyze the numbers to figure out what happens. We see that α_1 and α_2 are both negative. Let's redefine α_1 and α_2 to be $-\alpha_1$ and α_2:

$$\alpha_1 = \frac{r}{2m} + (\frac{r^2}{4m^2} - \omega_0^2)^{\frac{1}{2}}$$

$$\alpha_2 = \frac{r}{2m} - (\frac{r^2}{4m^2} - \omega_0^2)^{\frac{1}{2}}$$

$$\boxed{x = c_1 e^{-\alpha_1 t} + c_2 e^{-\alpha_2 t}}$$

This form makes it easier to see what's going on. We can see that both exponentials are decaying, and α_1 is larger than α_2, by looking at cases where $\frac{r^2}{4m^2} \gg \omega_0^2$. When $\alpha_1 > \alpha_2$, and we displace our spring A, we see that the first term will decay rapidly, and the second term will decay slower. This means that as time passes, the second term will dominate.

3.6 Critical Damping

Critical damping is the case where

$$\frac{r^2}{4m^2} = \omega_0^2$$

In this case, $\alpha_1 = \alpha_2 = \frac{r}{2m} = \alpha$, we are left with:

$$x_1 = ce^{-\alpha t}$$

We have a second order diffeq, so we need another solution, whichis

$$x_2 = c_2 te^{-\alpha t}$$

To demonstrate that this is also a solution, we take the case when $\alpha_1 \to \alpha_2$:

$$x = c_1 e^{-\alpha_1 t} + c_2 e^{-\alpha_2 t}$$

As $\alpha_1 - \alpha_2 \to 0$:

$$x = e^{-\alpha_1 t}(c_1 + c_2 e^{-(\alpha_2 - \alpha_1)t})$$

We see that the interior exponential is approximately a constant:

$$x \approx e^{-\alpha t}(c_1 + c_2(-\alpha_2 + \alpha_1)t)$$

Critically damped systems essentially have a faster falloff compared to overdamped systems, because

$$\frac{r}{2m} > \frac{\omega_0^2 m}{r}$$

In general, the critical damping solution is of the form:

$$\boxed{x(t) = (A + Bt)e^{-\omega t}}$$

4 Forced Harmonic Oscillation

We now want to introduce a driving force to our simple mass on a spring system. If we take x to be the left/right axis, we can look at the forces on the mass. The first is the restoring force of the spring:

$$F = -sx$$

We also have the dissipative/damping force:

$$F = -sx - r\frac{dx}{dt}$$

And we now have a driving force:

$$F = -sx - r\frac{dx}{dt} + F\cos(\omega t) = ma$$

Rewriting the system:

$$m\frac{d^2x}{dt^2} + r\frac{dx}{dt} + sx = F\cos(\omega t)$$

Note that the ω we are using here is different from $\omega_0 = \sqrt{\frac{s}{m}}$ (fundamental frequency of the system, also known as the normal mode). We want to see how this system behaves, and how to analyze this system. Let's now make an attempt to solve the problem.

4.1 Solving the Driven Oscillator

$$m\frac{d^2x}{dt^2} + r\frac{dx}{dt} + sx = F\cos(\omega t)$$

We start by using complex algebra to change the cosine into an exponential:

$$F\cos(\omega t) = Re(Fe^{i\omega t})$$

However, we disregard taking the real portion until after we've solved for x, and we just use the entire exponential:

$$m\frac{d^2x}{dt^2} + r\frac{dx}{dt} + sx = Fe^{i\omega t}$$

To solve this diffeq, we assume $x = Ae^{i\omega t}$. We can then put this into the equation:

$$\dot{x} = i\omega Ae^{i\omega t}$$

$$\ddot{x} = -\omega^2 Ae^{i\omega t}$$

Inserting these into the equation:

$$m(-\omega^2)Ae^{i\omega t} + r(i\omega)Ae^{i\omega t} + sAe^{i\omega t} = Fe^{i\omega t}$$

We can cancel out the exponentials:

$$-mA\omega^2 + irA\omega + sA = F$$

Doing some algebra and substituting in $\omega_0^2 = \frac{s}{m}$ and defining $\gamma = \frac{r}{m}$:

$$A = \frac{F_0}{m}\frac{1}{(\omega_0^2 - \omega^2) + i\omega\gamma}$$

However, this isn't satisfactory, since the want a real number for the amplitude. Currently, A is complex valued:

$$A = |A|e^{i\phi}$$

We can multiply by the complex conjugate of the denominator:

$$A = \frac{F_0}{m}\frac{(\omega_0^2 - \omega^2) - i\omega\gamma}{(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2}$$

We get that $|A|^2 = AA^*$, and we see that this is real valued:

$$\boxed{|A|^2 = \frac{F_0^2}{m^2}\frac{1}{(\omega_0^2-\omega^2)^2+\omega^2\gamma^2}}$$

The textbook factors out a $-i$ from A:

$$A = \frac{-iF_0}{m}\frac{\omega\gamma + i(\omega_0^2-\omega^2)}{(\omega_0^2-\omega^2)+\omega^2\gamma^2}$$

From this we can get something that relates to the phase:

$$\boxed{\tan\phi = \frac{\omega^2-\omega_0^2}{\omega\gamma}}$$

This is very important, as resonances happen everywhere, in mechanical systems, atomic physics, particle physics, and many other systems.

If we look at just $|A|$:

$$|A| = \frac{F_0}{m}\frac{1}{\sqrt{(\omega_0^2-\omega^2)^2+\omega^2\gamma^2}}$$

We can break this down into different conditions. If we have $\omega = 0$, we can get the amplitude:

$$\frac{F_0}{m}\frac{1}{\omega_0^2} = \frac{F_0}{m}\frac{1}{\frac{s}{m}} = \frac{F_0}{s}$$

In the case where $\omega \to \infty$, the amplitude approaches 0.

In the case where $\omega = \omega_0$:

$$A = \frac{F_0}{m}\frac{1}{\omega_0\gamma} = \frac{F_0}{m}\frac{m}{\omega_0 r}$$

Recall the definition $Q = \frac{\omega_0 m}{r}$, and using this we can rewrite A:

$$\boxed{A = \frac{F}{\omega_0^2 m}Q = \frac{F}{s}Q}$$

This is what we call resonance. If we graph A, we can see that it is not symmetric, and it peaks at $\omega = \omega_0$, and the larger the value of Q, the higher the peak.

If we look at our solution, where we express A in terms of its magnitude and its phase:

$$x = \frac{-iF}{m}\frac{e^{i\phi}}{\sqrt{(\omega_0^2-\omega^2)^2+\omega^2\gamma^2}}e^{i\omega t}$$

We want to take the real part of this, so we merge the exponentials and expand into sines and cosines, leaving us with:

$$Re(x) = |A|\sin(\omega t - \phi)$$

If we compare the relative phase between x and the driving force, we see that at $\omega = \omega_0$ (resonance), the phase difference between the displacement and the driving force is exactly $\frac{\pi}{2}$.

4.2 Energy of Driven Oscillation

How does the energy of this system behave as a function of ω? We need to know that

$$E \propto |A|^2$$

This tells us that

$$E \propto (\frac{F_0}{m})^2 \frac{1}{(\omega_0^2 - \omega^2) + \omega^2\gamma^2}$$

If we had to draw the energy as a function of frequency, we would once again see that it peaks at $\omega = \omega_0$, and goes to 0 at ∞. The fact that the energy peaks at the natural frequency means that the most efficient way of inputting energy into the system is at $\omega = \omega_0$. We now want to ask where this energy goes. The driving energy inputs energy, and the dissipative force removes energy from the system. This means that at the natural frequency, we have the maximum dissipation of energy, as well as the maximum amount of energy input (because the net energy change has to be 0).

We have shown that the driven oscillator has a solution that takes the form $Ae^{i\omega t}$, and that A is a complex number with some amplitude and phase. We can get the amplitude via $|A|^2 = AA^*$, and we see that it depends on γ and the frequency ω. We also know that since A is complex, it has a phase, which we computed and we showed that the phase can be written:

$$\tan\phi = \frac{\omega^2 - \omega_0^2}{\omega\gamma}$$

We also showed that we can get the real solution by taking the real portion of the solution we had:

$$Re(x) = |A|\sin(\omega t - \phi)$$

We also showed that if we had a static force, $A = \frac{F_0}{s}$. We also showed that if we used the definition of quality Q, we see that we can rewrite A:

$$A = \frac{F_0}{\omega_0^2 m} Q$$

We then showed that if we plotted the amplitude as a function of frequency, we see that it peaked at $\omega = \omega_0$, and the higher the Q, the higher the peak.

We then also discovered that we have a phase difference between the force $F_0\cos(\omega t)$, and the displacement $A\sin(\omega t - \phi)$ (which comes from the fact that we have a cosine and a sine, which are offphase by $\frac{\pi}{2}$) of $\Delta\phi = \phi + \frac{\pi}{2}$. We can see that at resonance, the phase difference between the force and the displacement is $\frac{\pi}{2}$. These apply to almost every resonance that we will see in physics.

We then looked at the energy, and we saw that $E \propto |A|^2$. We once again noticed that this peaked at $\omega = \omega_0$.

4.3 Resonance Peaks

We now want to know how wide the peak is. We know that the peak is centered at ω_0, the natural frequency, and we define the Full Width Half Maximum (FWHM). This means we are looking for the frequency at which the energy is at 1/2 of the energy at the peak. The energy is proportional to

$$\frac{F_0^2}{m^2}\frac{1}{\omega^2\gamma^2}$$

And we claim that FWHM occurs when

$$(\omega_0^2 - \omega^2)^2 = \omega^2\gamma^2$$

This gives us the position of the half peaks are at

$$\omega_0^2 - \omega^2 = \pm\omega\gamma$$

This gives us our two points, one with the $+$, and one with the $-$. If we do some algebra, we find that

$$\omega_2 - \omega_1 = \gamma = \frac{r}{m}$$

This tells us that the width of the peak is given by γ. This makes sense, since the smaller the r, the narrower the peak, and also Q increases, which gives us a narrow tall peak, exactly what we expect.

If we call $\omega_2 - \omega_1$ $\Delta\omega$, we can rewrite Q:

$$Q = \frac{\omega_0}{\Delta\omega}$$

We can see that we can define any resonance peak just given the peak frequency ω_0, and the Q value at the peak.

4.4 Energy in the System

4.4.1 Removing the Driving Force

What happens if we turn off the driving force? Let's assume that we have an underdamped system. We can see that we have some stored energy E_0. If we look at this as a function of time, we can see that it decays like

$$\begin{aligned} E &= E_0 e^{-\frac{r}{m}t} \\ &= E_0 e^{-\gamma t} \end{aligned}$$

We see that it is determined by the width of the resonance.

4.4.2 Power Delivered/Dissipated

Our system has 3 different forces, with the dissipative force removing energy from the system, the driving force increasing the energy in the system, and the restoring force not changing the energy in the system. If we have a circuit, the rate of the input power from the driving force has to be equal to the energy being removed by the dissipative force if we want to have an oscillation. Let's get the instantaneous power being added to the system:

$$\begin{aligned} P_I &= Fv \\ &= F_0 \cos(\omega t)\frac{dx}{dt} \\ &= F_0 A\omega \cos(\omega t)\cos(\omega t - \phi) \end{aligned}$$

We can break the last term up into its parts:

$$= \omega F_0 A \cos\omega t(\cos\omega t\cos\phi + \sin\omega t\sin\phi)$$

However, we're interested in the average power $\bar{P}$ (also seen as P_{avg}), on the scale of 1 single period.

$$\begin{aligned} P_{avg} &= \frac{1}{T}\int_0^T P_I\,dt \\ &= \frac{F_0 A\omega\cos\phi}{T}\int_0^T \cos^2\omega t\,dt + \frac{1}{T}\int_0^T \cos\omega t\sin\omega t\,dt \end{aligned}$$

We can see that the second term will always be 0, and the first integral is equal to $\frac{1}{2}$, giving us

$$\bar{P} = \frac{1}{2}F_0 A\omega\cos\phi$$

We know that $\tan\phi = \frac{\omega^2-\omega_0^2}{\gamma\omega}$, and we can plug this in and do some algebra and we get that

$$\boxed{\bar{P} = \frac{r}{2}\frac{F_0^2}{m^2}\omega^2\frac{1}{(\omega_0^2-\omega^2)^2+\gamma^2\omega^2}}$$

This is the expression for the power delivered by the driving force over one period.

Now we want to look at the power dissipated by $F_r = -r\dot{x}$. We claim that this is going to be exactly the same as the previous expression. We can do the math out:

$$\begin{aligned} P_I &= F_r v \\ &= rv^2 \\ &= r(\omega A\cos(\omega t-\phi))^2 \end{aligned}$$

Once again we are interested in the average power dissipated:

$$\begin{aligned} \bar{P} &= \frac{1}{T}\int_0^T P_I\,dt \\ &= \frac{1}{T}\int_0^T r\omega^2 A^2\cos^2(\omega t-\phi)\,dt \\ &= \frac{r\omega^2A^2}{T}\int_0^T \cos^2(\omega t-\phi)\,dt \\ &= \frac{r\omega^2A^2}{2} \end{aligned}$$

This tells us that the power dissipated by the dissipative force over 1 period is

$$\bar{P} = \frac{1}{2}r\omega^2A^2$$

If we now plug in the value of A:

$$\bar{P} = \frac{r}{2}\frac{F_0^2}{m^2}\omega^2\frac{1}{(\omega_0^2-\omega^2)^2+\gamma^2\omega^2)}$$

This shows us that the average power delivered and the average power dissipated are equal.

4.4.3 Transient State

Recall that the equation of motion that we are using takes the form

$$m\frac{d^2x}{dt^2} + r\frac{dx}{dt} + sx = F_0\cos(\omega t)$$

We can ask ourselves what would happen if we turned on the driving force later in the system? If we remove the driving force term, we see that we have a solution of the form

$$X_1 = A_1 e^{\frac{-r}{2m}t}\sin(\omega' + \phi_1)$$

where $\omega' = \sqrt{\omega_0^2 - \frac{r^2}{4m^2}}$. We also know that when we turn on the driving force, we have a solution of the form

$$X_2 = A_2\sin(\omega t - \phi_2)$$

This second solution is known as the steady state solution. We can make the claim that $X = X_1 + X_2$ is also a good solution of the equation of motion, which we can see just by inspection (Can think about diffeq and homogeneous solution and particular solution). This tells us that the general solution is

$$X = A_1 e^{-\frac{r}{2m}t}\sin(\omega' t + \phi) + A_2\sin(\omega t - \phi_2)$$

This solution means that if we start oscillating the system, the system will start off oscillating like a damped oscillation, and it will decay exponentially, leaving behind the steady state solution. This will give us something that looks like a system being driven by 2 frequencies (think about beats), with an inner oscillation and an enveloping oscillation.

Eventually this will die down, and we will be left with a steady state solution. The time before the system reaches the steady state is known as the transient state or transient mode.

4.5 RLC Circuits

An RLC circuit is made up of a capacitor, inductor, and a resistor in a closed circuit. In one of these circuits, the total voltage in the system is the driver. We get the equation of motion by taking the voltages of each of the components, V_L, V_R, and V_C, and summing them:

$$V_L + V_R + V_C = V_0\cos(\omega t)$$

Rewriting and replacing each term with the correct expression:

$$L\frac{dI}{dt} + IR + \frac{q}{c} = V_0\cos(\omega t)$$

4.5.1 Solving the Equation

We start by remembering that $I = \frac{dq}{dt}$, leaving us with

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{c} = V_0\cos(\omega t)$$

This looks exactly like the equation that we had for the spring. Traditionally, we don't solve it the same way. We first rewrite the RHS in exponential form, and then rewriting I as an exponential as well ($I = I_0 e^{i\omega t}$), and we solve the problem in terms of I instead of q. We can then compute some values:

$$V_L = i\omega L I_0 e^{i\omega t} \qquad V_R = RIe^{i\omega t} \qquad V_C = \frac{1}{c}\int I\,dt = \frac{1}{ic\omega} I_0 e^{i\omega t}$$

Inserting these all into the diffeq:

$$(i\omega L)I_0 e^{i\omega t} + RI_0 e^{i\omega t} - \frac{i}{c\omega} I_0 e^{i\omega t} = V_0 e^{i\omega t}$$

Factoring and solving for I_0:

$$I_0 = \frac{V_0}{R + i(L\omega - \frac{1}{c\omega})}$$

This quantity in the denominator is known as Z, and is called the impedance. This quantity is complex:

$$Z = |Z|e^{i\phi}$$

$$|Z| = (R^2 + (L\omega - \frac{1}{c\omega})^2)^{\frac{1}{2}}$$

$$\tan\phi = \frac{L\omega - \frac{1}{c\omega}}{R}$$

If we go back to I:

$$I = \frac{V_0 e^{i(\omega t - \phi)}}{(R^2 + (L\omega - \frac{1}{c\omega})^2)^{\frac{1}{2}}}$$

Taking the real part of this:

$$\boxed{Re(I) = \frac{V_0}{|Z|}\cos(\omega t - \phi)}$$

4.5.2 Voltage Across Inductor/Resistor/Capacitor

If we have Z and I_0, we can find quantities like V_R:

$$V_R = RI = R(\frac{V_0}{|Z|})e^{i(\omega t - \phi)}$$

Taking the real part:

$$Re(V_R) = \frac{R}{Z} V_0 \cos(\omega t - \phi)$$

We can see that if we compare the phase of V and R, we have a phase difference of ϕ.

Doing the same process for V_L:

$$V_L = i^2 L\omega = \frac{L\omega}{Z} V_0 (ie^{i(\omega t - \phi)})$$

This i on the outside rotates it by $\pi/2$:

$$V_L = \frac{L\omega}{|Z|} V_0 e^{i(\omega t - \phi + \frac{\pi}{2})}$$

Taking the real part:

$$Re(V_L) = \frac{L\omega}{|Z|} V_0 \cos(\omega t - \phi + \frac{\pi}{2})$$

Doing the same thing for V_C:

$$V_C = \frac{-i}{c\omega} I = \frac{\frac{1}{c\omega}}{|Z|} V_0(-ie^{i(\omega t-\phi)})$$

Using the fact that $-i = e^{-\frac{\pi}{2}}$:

$$V_C = \frac{1}{c\omega|Z|} V_0 e^{i(\omega t-\phi-\frac{\pi}{2})}$$

Taking the real portion:

$$Re(V_C) = \frac{1}{c\omega|Z|} V_0 \cos(\omega t - \phi - \frac{\pi}{2})$$

Notice that V_R peaks at $\omega = \omega_0$, but V_C and V_L peak at different times. Also note that the natural frequency is $\frac{1}{\sqrt{LC}}$.

If we take V_R:

$$V_R = \frac{RV_0 \cos(\omega t - \phi)}{Z}$$

$$V_R = \frac{RV_0 \cos(\omega t - \phi)}{\frac{L}{\omega}((\frac{\omega R}{L})^2 + (\omega_0^2 - \omega^2)^2)^{\frac{1}{2}}}$$

If we plot this, we see that the peak is close to ω_0 for large Q:

$$Q = \frac{\omega_0 L}{R}$$

We can again work out the instantaneous power:

$$P = VI$$

and we can then calculate the average power:

$$\bar{P} = \frac{1}{T}\int_0^T VI\,dt$$

We can see that this peaks at $\omega = \omega_0$, and the FWHM for the power curve is given by $\frac{R}{L}$.

5 Normal Modes and Coupled Oscillators

If we think about an atomic lattice, the bonds between the atoms are electromagnetic, and we can actually think of them as being bound by springs. This is a very classical solution to an inherently quantum problem, but it turns out that the classical solution gives some insight into how the lattice behaves. We could find out what type of vibration frequencies can actually propagate through the material. We will use these coupled oscillators to learn about how waves propagate through a medium.

Taking the example of the atomic lattice again, if we disturb just one atom in the lattice (let's say its just a 2d lattice for now). We can think of this setting off a wave throughout the rest of the lattice.

5.1 Coupled Spring and Mass Oscillators

We can start from a simple system, two masses connected via three springs to two walls. For simplicity's sake, we'll keep the masses equal to m, and the spring constants of the 2 side springs equal to k. We call the spring constant of the center spring k'. We can call the distance from the equilibrium position of the two masses, x_1 and x_2, which both start at 0. To start the oscillation, we can move the second mass some amount ($x_2 = A$).

If we want to find out what the force acting on mass 1 is:

$$F_1 = m\ddot{x}_1 = m\frac{d^2x_1}{dt^2}$$

And the force acting on mass 2:

$$F_2 = m\ddot{x}_2 = m\frac{d^2x_2}{dt^2}$$

Looking at all the forces acting on the first spring:

$$F_1 = -kx_1 - k'(x_1 - x_2)$$

And the forces acting on the second spring:

$$F_2 = -kx_2 - k'(x_2 - x_1)$$

Note that the second term has to be opposite because of Newton's third law. However, this method of finding the forces is kind of handwavy, so we have a better way of solving the problem.

5.1.1 A Better Way to Find the Forces

We want to use the potential energy ($F = -\frac{dU}{dx}$). We know it will be the sum of the potential energies of the springs:

$$U = \frac{1}{2}kx_1^2 + \frac{1}{2}k'(x_1 - x_2)^2 + \frac{1}{2}kx_2^2$$

If we now want to get the force F_1, we can take the partial derivative with respect to x_1:

$$\begin{aligned} F_1 &= -\frac{\partial U}{\partial x_1} \\ &= -kx_1 - k'(x_1 - x_2) \end{aligned}$$

This is exactly what we got before. We can do the same thing for F_2, taking the partial derivative of the potential with respect to x_2, and we will see that we get what we got prior to this. This method is almost always viable, so its generally a better choice than thinking about the forces from the start.

Now that we have these two equations, we can use $F = ma$ to get the **coupled differential equations**:

$$\boxed{m\frac{d^2x_1}{dt^2} = -kx_1 - k'(x_1 - x_2) \qquad m\frac{d^2x_2}{dt^2} = -kx_2 - k'(x_2 - x_1)}$$

5.1.2 Solving the Coupled Equations

One method to solve this is to sum the two equations, and subtract the two equations, and we see that terms will cancel. Let's think about this system physically. One type of motion is when the two masses are moving in the same direction, and we can see that the oscillations work together. The other type of motion is the right mass going right and the left mass going to the left. This is when they are oscillating separately/opposite of each other.

Adding the two equations:

$$m(\frac{d^2x_1}{dt^2} + \frac{d^2x_2}{dt^2}) = -k(x_1 + x_2)$$

And subtracting the two equations:

$$m(\frac{d^2x_1}{dt^2} - \frac{d^2x_2}{dt^2}) = -(k + 2k')(x_1 - x_2)$$

Taking the top equation, we can call $z = x_1 + x_2$, and we are left with

$$m\frac{d^2z}{dt^2} = -kz$$

We can see that this will just be simple harmonic motion:

$$(x_1 + x_2) = 2A_s \cos(\omega_s t + \phi_s)$$

We can see that $\omega_s = \sqrt{\frac{k}{m}}$. This is a slow oscillation.

Looking at the second equation, we can call $z' = x_1 - x_2$:

$$m\frac{d^2z'}{dt^2} = -(k + 2k')z'$$

This is also simple harmonic motion, with frequency $\omega_f = \sqrt{\frac{k+2k'}{m}}$:

$$(x_1 - x_2) = 2A_f \cos(\omega_f t + \phi_f)$$

We use the 2s because they will help us later.

5.1.3 Normal Modes

The system supports two modes of oscillation, ω_s and ω_f. We have what we call the normal coordinates/variables, $x_1 + x_2$ and $x_1 - x_2$. Note that if the two masses were different, we would have normal variables that are different, and that are dependent on the masses. However, we can solve that issue with the same methods that we used here. If we write down the solutions:

$$(x_1 + x_2) = 2A_s \cos(\omega_s t + \phi_s) \qquad (x_1 - x_2) = 2A_f \cos(\omega_f t + \phi_f)$$

If we take the sum of these and divide by 2:

$$x_1 = A_s \cos(\omega_s t + \phi_s) + A_f \cos(\omega_f t + \phi_f)$$

If we subtract and divide by 2:

$$x_2 = A_s \cos(\omega_s t + \phi_s) - A_f \cos(\omega_f t + \phi_f)$$

5.1.4 Example of a Special Case

Assume that $\dot{x}_1 = 0$ and $\dot{x}_2 = 0$ for initial conditions, and $x_1 = 0$, $x_2 = A$. Looking at the solutions:

$$x_1 = A_s \cos(\omega_s t + \phi_s) + A_f \cos(\omega_f t + \phi_f)$$

Taking the derivative:

$$\dot{x}_1 = -\omega_s A_s \sin(\omega_s t + \phi_s) - \omega_f \sin(\omega_f t + \phi_f)$$

Plugging in initial conditions:

$$\dot{x}_1(0) = -\omega_s A_s \sin\phi_s - \omega_f A_f \sin\phi_f = 0$$

Repeating this process for x_2:

$$\dot{x}_2(0) = -\omega_s A_s \sin\phi_s + \omega_f A_f \sin\phi_f = 0$$

Adding the two expressions, we get that $\sin\phi_s = 0$, and if we subtract them, we get $\sin\phi_f = 0$. We take the simplest condition that $\phi_s = \phi_f = 0$.

For the amplitudes:

$$x_1(t) = A_s \cos\omega_s t + A_f \cos\omega_f t$$
$$x_1(0) = A_s + A_f = 0$$
$$A_s = -A_f$$

Doing the same thing for x_2, we get that $A_s - A_f = A$, which gives us that $A_s = -A_f = \frac{A}{2}$. We can then rewrite our solutions:

$$x_1(t) = \frac{A}{2}(\cos\omega_s t - \cos\omega_f t) \qquad x_2(t) = \frac{A}{2}(\cos\omega_s t + \cos\omega_f t)$$

If we recall the lecture on beats:

$$x_1(t) = A\sin(\frac{\omega_f - \omega_s}{2}t)\sin(\frac{\omega_f + \omega_s}{2}t) \qquad x_2(t) = A\cos(\frac{\omega_f - \omega_s}{2}t)\cos(\frac{\omega_f + \omega_s}{2}t)$$

We have talked about how to analyze a chain of coupled oscillators, primarily a system with 3 springs and 2 masses. We figured out how to find the forces associated with the system, and we were able to get the equations of motion that governed the system. We saw that they were coupled differential equations, meaning that there was no easy way of solving them. We saw that if we added/subtracted the equations, we were left with self contained differential equations, which we saw gave us two different oscillation types, one being the fast oscillation and the other being the slow oscillation.

5.1.5 Determinant Method

We can actually solve this set of couple equations in another method, known as the determinant method, which is more cumbersome, but works better in general.

$$m\frac{d^2x_1}{dt^2} = -kx_1 - k'(x_1 - x_2) \qquad m\frac{d^2x_2}{dt^2} = -kx_2 - k'(x_2 - x_1)$$

We say that $x_1 = A_1 e^{i\omega t}$, and that $x_2 = A_2 e^{i\omega t}$. We write this in a matrix format:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} e^{i\omega t}$$

Taking these solutions and plugging them into the coupled equations:

$$-m\omega^2 A_1 e^{i\omega t} = (-kA_1 - k'A_1 + k'A_2)e^{i\omega t}$$

$$-m\omega^2 A_2 e^{i\omega t} = (-kA_2 - k'A_2 + k'A_1)e^{i\omega t}$$

We can see that the exponentials cancel out in both equations:

$$-m\omega^2 A_1 = (-kA_1 - k'A_1 + k'A_2)$$

$$-m\omega^2 A_2 = (-kA_2 - k'A_2 + k'A_1)$$

And rewriting, moving everything to one side:

$$(-m\omega^2 + k + k')A_1 - k'A_2 = 0$$

$$(-m\omega^2 + k + k')A_2 - k'A_1 = 0$$

Writing this in matrix form:

$$\begin{bmatrix} -m\omega^2 + k + k' & -k' \\ -k' & -m\omega^2 + k + k' \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = 0$$

To solve this, we see that we have the trivial solution $A_1 = 0$, $A_2 = 0$, but we also have a non-trivial solution. The non-trivial solution will occur in situations where our matrix doesnt have an inverse, which via linear algebra is true when the determinant of the matrix is 0. The determinant of this matrix is

$$(-m\omega^2 + k + k')^2 - k'^2 = 0$$

If we solve this, we find that we have two solutions:

$$\omega_s = \sqrt{\frac{k}{m}} \qquad \omega_f = \sqrt{\frac{k + 2k'}{m}}$$

These are the same solutions that we had before. We can also plug these back into the equations to get the same normal coordinates that we got before.

5.1.6 Adding in Damping/Driving Forces

If we added a dissipative force to both masses of $F_d = -r\dot{x}$, and a driving force $F = F_0 \cos\omega t$ to mass 1, what would our equations look like?

$$m\frac{d^2x_1}{dt^2} = -kx_1 - k'(x_1 - x_2) - r\dot{x}_1 - F_0 \cos\omega t$$

$$m\frac{d^2x_2}{dt^2} = -kx_2 - k'(x_2 - x_1) - r\dot{x}_2$$

Taking the sum:

$$m(\ddot{x}_1 + \ddot{x}_2) = -k(x_1 + x_2) - r(\dot{x}_1 + \dot{x}_2) + F_0 \cos\omega t$$

and taking the difference:

$$m(\ddot{x}_1 - \ddot{x}_2) = -(k + 2k')(x_1 + x_2) - r(\dot{x}_1 - \dot{x}_2) + F_0 \cos \omega t$$

Taking these equations, we can see that we'd have two resonances:

$$\omega_s = \sqrt{\frac{k}{m}} \qquad \omega_f = \sqrt{\frac{k + 2k'}{m}}$$

And we'd have

$$x_1 + x_2 = 2A_s \cos(\omega_s t + \phi_s)$$

Where

$$A_s = \frac{F_0}{\sqrt{(\omega_s^2 - \omega^2)^2 + (\frac{r}{m}\omega)^2}}$$

And something similar for A_f:

$$A_f = \frac{F_0}{\sqrt{(\omega_f^2 - \omega^2)^2 + (\frac{r}{m}\omega)^2}}$$

You could then do the same thing we did before, and get solutions for x_1 and x_2.

5.2 Pendulums with Spring

Take the system comprising of 2 masses m as pendulums, connected by a spring of spring constant k. If we disturb the system, there are several things that can happen. In one situation, they can swing together, and in another, they swing in opposite directions. This tells us that we have 2 normal modes. Lets get the equations of motion by getting the potential energy:

$$U = \frac{1}{2}k(x_1 - x_2)^2 + \frac{1}{2}mg\frac{x_1^2}{l} + \frac{1}{2}mg\frac{x_2^2}{l}$$

Taking the partial derivatives:

$$F_1 = -\frac{\partial U}{\partial x_1} = -k(x_1 - x_2) - \frac{mgx_1}{l}$$
$$F_2 = -\frac{\partial U}{\partial x_2} = -k(x_2 - x_1) - \frac{mgx_2}{l}$$

Implementing $F = ma$, and adding/subtracting the two equations:

$$m\frac{d^2(x_1 + x_2)}{dt^2} + \frac{g}{l}(x_1 + x_2) = 0$$

$$m\frac{d^2(x_1 - x_2)}{dt^2} + ((\frac{g}{l}) + 2\frac{s}{m})(x_1 - x_2)$$

We once again see that we have the same two normal coordinates, $x_1 + x_2$ and $x_1 - x_2$.

5.3 Three Mass System

What if we took the first system, but added another mass and spring (and made all the spring constants the same)? In this system, using the potential method to get the forces helps a lot:

$$U = \frac{1}{2}kx_1^2 + \frac{1}{2}k(x_1 - x_2)^2 + \frac{1}{2}k(x_2 - x_3)^2 + \frac{1}{2}kx_3^2$$

We can then go ahead and take the partial derivatives. We will then see that there is no easy method of adding/subtracting the equations, so the best approach is to use the determinant/matrix method. We see that we'll get 3 normal modes, 3 normal coordinates, and 3 normal frequencies.

5.4 Loaded Strings

We want to look at systems that are loaded strings, strings under tension with masses at equal distances placed on top of them. We want to see what happens if we displace one of the masses vertically. Intuitively, the mass will start to oscillate, and the oscillation will cause the masses to either side to oscillate. This will create a wave of oscillations of the masses along the string.

If we look at some arbitrary mass along the string (the masses are apart by a distance of a), and we raise it by a height of y, we see that we have two tensions, one to either side. We can name the angles from the horizontal to the string, θ_1 and θ_2, and we see that we have a vertical restoring force (the horizontal tension forces cancel each other out):

$$-T(\sin\theta_1 + \sin\theta_2)$$

Now we can try to find expressions for $\sin\theta_1$ and $\sin\theta_2$. We label the masses via r's, and we have the height difference between the left bead $r-1$ and the bead being lifted r, and we can get an expression:

$$\sin\theta_1 = \frac{y_r - y_{r-1}}{a}$$

And one for the angle on the right side:

$$\sin\theta_2 = \frac{y_r - y_{r+1}}{a}$$

Inserting these into the expression and using $F = ma$:

$$\boxed{m\frac{d^2y_r}{dt^2} = \frac{T}{a}(y_{r-1} - 2y_r + y_{r+1})}$$

We have an equation of motion for mass number r, which we can now try to solve. We can now guess a solution:

$$y_r = A_r e^{i\omega t}$$

where $A_r \in \mathbb{C}$. Doing the math and inserting this guess into the equation, we find that

$$\boxed{-A_{r-1} + (2 - \frac{ma\omega^2}{T})A_r - A_{r+1} = 0}$$

Let's now impose some boundary conditions. We assume that the first and last beads are tied down, so $y_0 = A_0 = 0$ and $y_{n+1} = A_{n+1} = 0$. If we do this, and we say that $n = 1$, we find that

$$(2 - \frac{ma\omega^2}{T})A_1 = A_2$$

For this to be true, the left term must be 0, so

$$\omega_1^2 = \frac{2T}{ma}$$

We have n possible ωs, labelled 1 through n. We call $\frac{T}{ma} = \omega_0^2$, so we see that $\omega_1^2 = 2\omega_0^2$.

Let's go back to the general solution, and lets rewrite it:

$$\begin{aligned}
-A_{r-1} + (2 - \frac{ma\omega^2}{T})A_r - A_{r+1} = 0 \\
\frac{A_{r-1} + A_{r+1}}{A_r} &= (2 - \frac{ma\omega^2}{T}) \\
&= \frac{2\frac{T}{ma} - \omega^2}{\frac{T}{ma}} \\
&= \frac{2\omega_0^2 - \omega^2}{\omega_0^2} \\
\frac{A_{r-1} + A_{r+1}}{A_r} &= \frac{2\omega_0^2 - \omega^2}{\omega_0^2}
\end{aligned}$$

This equation is interesting because the right side of the equation is totally independent of r.

Let's now look at A_r. We have said that it is a complex quantity, which has a phase. Via intuition, we can say that the phase of nearby masses will be incrementally different from each other. We can write A_r:

$$A_r = Ce^{ir\theta}$$

where $C = |C|e^{i\delta}$. Note that δ is the phase that is global, and is in all the masses, while θ changes depending on the mass. If we put this into practice in the equation above:

$$\begin{aligned}
\frac{Ce^{i(r-1)\theta} + Ce^{i(r+1)\theta}}{Ce^{ir\theta}} &= \\
&= e^{-t\theta} + e^{i\theta} \\
&= 2\cos\theta
\end{aligned}$$

Inserting this and looking at the right side:

$$2\cos\theta = \frac{2\omega_0^2 - \omega^2}{\omega_0^2}$$

Now lets apply the boundary conditions. We know that $A_0 = 0$:

$$Re(A_0) = |C|\cos\delta = 0$$

This gives us that $\delta = \frac{\pi}{2}$. This means that $A_r = |C|e^{ir(\theta + \frac{\pi}{2})}$. Taking the real portion:

$$Re(A_r) = |C|\sin r\theta$$

Applying the boundary condition that $A_{n+1} = 0$:

$$Re(A_{n+1}) = |C|\sin(n+1)\theta = 0$$

This means that $(n+1)\theta = j\pi$, where $j = 1, 2, \ldots, n$. Note that the different values for j represent the n normal modes. Solving for θ:

$$\theta = \frac{j\pi}{n+1}$$

Plugging this in:

$$A_r = |C| \sin(\frac{rj\pi}{n+1})$$

In this expression, n is the number of masses, r is the mass number, and j is the normal mode of oscillation. We can get an expression for the amplitude of the rth mass in normal mode j:

$$\boxed{y_r^j(t) = |C| \sin(\frac{rj\pi}{n+1}) e^{i\omega_j t}}$$

Where ω_j can be given via

$$\omega_j^2 = 2\omega_0^2(1 - \cos(\frac{j\pi}{n+1}))$$

and $\theta_j = \frac{j\pi}{n+1}$. We see that we have a system with a maximum frequency of $2\omega_0$. We know that $\omega_0^2 = \frac{T}{ma}$:

$$\omega_0^2 = \frac{1}{a^2}\frac{T}{\mu}$$

where μ is the mass density (kg/m). If we go back and look at the NaCl atomic lattice, we don't really know what T is, but we do know the mass density μ. If we take a saly crystal and apply some force F, we can look at the Young Modulus:

$$Y = \frac{\frac{F}{A}}{\frac{\Delta L}{L}}$$

where A is area. For salt, we see that this value is 10^{11} Newtons per square meters. If we take this value and divide it by ρ (volume density), we can see that this has the same units as T/μ. We make the assumption that

$$2\omega_0 \approx \frac{1}{a}\sqrt{\frac{Y}{\rho}}$$

Taking the atomic spacing for salt, and inserting everything in, we see that $2\omega_0$ for salt is roughly 10^{13} Rad/s, which is in the infrared spectrum. The maximum frequency $2\omega_0$ is know as the cutoff frequency. If we think about oscillating the first mass on the string, we can physically see that the oscillations propagate like a wave through the rest of the string. We have that

$$\frac{d^2 y_r}{dt^2} = \frac{T}{ma}(r_{r+1} - 2y_r + y_{r-1})$$

And we can think about what happens if we make the spacing a very small, making it some dx. We can now label each mass from just the x position along the string. Rewriting this formula:

$$\frac{\partial^2 y(x,t)}{\partial t^2} = \frac{T}{m}(\frac{y_{r+1} - y_r}{a} - \frac{y_r - y_{r-1}}{a})$$

Note that we Taylor expanded $y_{r+1} = y(x+\Delta x) = y(x) + dx\frac{dy}{dx} + \frac{1}{2}(dx)^2\frac{d^2y}{dx^2}$, and $y_{r-1} = y(x-\Delta x) = y(x) - dx\frac{dy}{dx} + \frac{1}{2}(dx)^2\frac{d^2y}{dx^2}$. Inserting these into the expression and simplifying:

$$\frac{\partial^2 y(x,t)}{\partial t^2} = \frac{T}{m}(dx\frac{d^2y}{dx^2})$$

Recall that mass density $\mu = \frac{m}{a} = \frac{m}{dx}$, so we are left with

$$\boxed{\frac{\partial^2 y(x,t)}{\partial t^2} = \frac{T}{\mu}\frac{\partial^2 y}{\partial x^2}}$$

This is telling us that if we have a string tied at both ends, and extremely close masses along the entire string, and we disturb one of the masses, this function will describe how the disturbance from the start point will travel through the string. This is what we call a wave, a propagation of a disturbance in a string. If we want to find the speed of the wave, we can do some dimensional analysis. Looking at $\frac{T}{\mu}$, we see that it has dimensions of velocity squared. We claim that this has the right properties to be the speed of the wave, but that's just a guess.

6 Transverse Waves

We want to look at transverse waves in a string that is under tension. We say that the string is under tension T. We take a tiny portion of the string, of length δx. If we disturb it by a slight angle θ from the horizontal, we have a transverse force $T\sin\theta$. The restoring force is $T\sin(\theta+\delta\theta) - T\sin\theta$. If θ is small, then $\sin\theta \approx \tan\theta$, which is approximately equal to the slope of the string, which is $\frac{dy}{dx}$. The restoring force can be written as

$$\begin{aligned} F_R &= T(\frac{dy(x+\Delta x)}{dx} - \frac{dy(x)}{x}) \\ &= T\frac{d^2y(x)}{dx^2}\delta x \end{aligned}$$

By Newton's Law $F = ma$ (Also realizing that we're actually using partials since y depends on t):

$$T(\frac{\partial^2 y(x,t)}{\partial x^2}\delta x) = m\frac{\partial^2 y(x,y)}{\partial t^2}$$

We know that $m = \rho\delta x$, where ρ is mass per unit length.

$$T(\frac{\partial^2 y(x,t)}{\partial x^2})\delta x = \rho\delta x\frac{\partial^2 y(x,t)}{\partial t^2}$$

Cancelling δx, we discover that

$$\boxed{\frac{\partial^2 y(x,t)}{\partial x^2} = \frac{T}{\rho}\frac{\partial^2 y(x,t)}{\partial t^2}}$$

This is just the same type of wave equation we found earlier. We see that T/ρ has units of velocity squared. Throughout this course, we will see many instances where we end up with a differential equation that looks like this, the solution of which describes the propagation of a wave. For example, we will work with longitudinal waves (like sound waves). We want to come up with an equation that looks like this. The same applies to electromagnetic systems, like waves travelling through a cable. We will use Maxwell's equations to get a wave equation that looks like this.

6.1 Solution to the Wave Equation

We want to find the solution to the general wave equation

$$\frac{\partial^2 y(x,t)}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y(x,t)}{\partial t^2}$$

It can be shown that any function of the form $y = f(ct - x)$ is a solution to this. If we let $z = ct - x$, and we compute the derivatives:

$$\frac{\partial y}{\partial x} = \frac{\partial f}{\partial z}\frac{\partial z}{\partial x} = -f'$$

$$\frac{\partial^2 y}{\partial x^2} = f''$$

$$\frac{\partial y}{\partial t} = cf' \qquad \frac{\partial^2 y}{\partial t^2} = c^2 f''$$

Inserting these into the wave equation:

$$f''(ct - x) = \frac{1}{c^2}(c^2 f''(ct - x))$$

$$f''(ct - x) = f''(ct - x) \quad \square$$

We can also similarly show that $y = f(ct + x)$ is also a viable solution. Let's think about what the physical meaning of this function is. If we factor the expression:

$$y = f(ct - x) = f(c(t - \frac{x}{c}))$$

This means that if we take the string at location x, and we knew the initial conditions of the string at x, this is the same thing as what the string is doing at $t = t - \frac{x}{c}$. In other words, the transverse position of the string at (x, t) is the same as that at $t - \frac{x}{c}$. This means that the wave has a finite velocity. The disturbance in a physical wave is never instantly propagated. This specific type of wave is called a right travelling wave, and $y = f(ct + x)$ is a left moving/travelling wave. In general, you can have both left moving and right moving waves in a string, so we have a general solution:

$$y = f_1(ct - x) + f_2(ct - x)$$

If we assume that the initial disturbance is SHM:

$$y(x = 0, t) = a \sin \omega t$$

then the solution to the wave equation would be

$$y = a \sin(\frac{\omega}{c}(ct - x))$$

We can rewrite ω/c:

$$\frac{\omega}{c} = \frac{2\pi v}{c} = \frac{2\pi}{cT} = \frac{2\pi}{\lambda}$$

where λ is known as the wavelength, the distance between one crest to crest or trough to trough. Note that this all only works if we have the disturbance being a periodic function.

We said that

$$y = a\sin(\frac{\omega}{c}(ct - x))$$

Which we can rewrite:

$$y = a\sin(\omega t - \frac{\omega x}{c})$$

We can define $k = \frac{2\pi}{\lambda}$, which is known as the wave number:

$$y = a\sin(\omega t - kx)$$

Note that k has nothing to do with the notation sometimes used for spring constants. We can also write the equation exponentially:

$$y = ae^{i(\omega t - kx)}$$

6.2 Changing Mediums

Now we want to look at the string again, but in the case where the physical properties of the string change at some point. Say for example at some point along the string, the mass density changes from ρ_1 to ρ_2. We want to find out what happens to the transmission of the wave. We know that the velocity is given by $\sqrt{\frac{T}{\rho_1}}$ in the first part of the string, and in the second part of the string the wave has velocity $\sqrt{\frac{T}{\rho_2}}$.

What happens at the point where the medium changes? Let's zoom in on the one point, and call it $x = 0$. At this point, we need to get the boundary conditions. Before we do this, lets address one misconception of waves. When we look at the wave, particles aren't actually travelling from left to right, instead, the displacement is changing. Essentially, the forces on the string are varying, causing the wave.

If we have a continuous system, and we figure out the displacement from the initial wave y_i, we can think of some part of the wave being reflected back, and some of it being transmitted, y_r and y_t respectively. Continuity tells us that at the point $x = 0$:

$$y_i + y_r = y_t$$

We can also find the force acting on the string at that point:

$$T(\frac{\partial(y_i + y_r)}{\partial x}) = T(\frac{\partial y_t}{\partial x})$$

Looking at the boundary conditions, we know that

$$y_i = A_1 e^{i(\omega t - k_1 x)}$$

Note that k changes because it is dependent on ρ, which is changing. We also know that

$$y_r = Be^{i(\omega t + k_1 x)}$$

Both of these are in medium 1, with ρ_1, and note that y_r is left moving. The third part is

$$y_t = A_2 e^{i(\omega t - k_2 x)}$$

Applying the equations at $t = 0$ and $x = 0$, we see that

$$A_1 + B = A_2$$

Applying the second equation:

$$-k_1A_1 + k_1B = -k_2A_2$$

Using simple algebra, we see that the net result is that

$$\frac{A_2}{A_1} = \frac{2k_1}{k_1 + k_2}$$

$$\frac{B}{A_1} = \frac{k_1 - k_2}{k_1 + k_2}$$

We know what fraction of the wave reflects, and what fraction of the wave transmits through to the next medium. Rewriting, we see that

$$k_1 = \frac{\omega}{c_1} = \frac{\omega}{\sqrt{T}}\sqrt{\rho_1}$$

$$k_1 = \frac{\omega}{\sqrt{T}}\sqrt{\rho_2}$$

Using these:

$$\frac{A_2}{A_1} = \frac{2\sqrt{\rho_1}}{\sqrt{\rho_1} + \sqrt{\rho_2}}$$

$$\frac{B}{A_1} = \frac{\sqrt{\rho_1} - \sqrt{\rho_2}}{\sqrt{\rho_1} + \sqrt{\rho_2}}$$

Now lets consider some special cases. The first case is that of a brick wall. The string has some mass density ρ_1, and it reaches a wall, which is best represented as $\rho_2 = \infty$. If we do this, we see that the reflected wave

$$R = \frac{B}{A_1} = \frac{\sqrt{\rho_1} - \sqrt{\rho_2}}{\sqrt{\rho_1} + \sqrt{\rho_2}} = -1$$

We see that the entire wave gets reflected, and the wave is inverted. Looking at the transmitted wave

$$T = \frac{A_2}{A_1} = \frac{2\sqrt{\rho_1}}{\sqrt{\rho_1} + \sqrt{\rho_2}} = 0$$

This is exactly what we expected, no transmission through the brick wall.

Lets look at the case where $\rho_1 > \rho_2$, which means the first part of the string is much thicker than the second part of the string. Using the same equations, we see that

$$R = \frac{\sqrt{\rho_1} - \sqrt{\rho_2}}{\sqrt{\rho_1} + \sqrt{\rho_2}}$$

This tells us that $0 < R < 1$, and if we look at the expression for T, we see that $1 < T < 2$.

In the opposite case, where $\rho_2 > \rho_1$, we see that $-1 < R < 0$, and that $0 < T < 1$. We see that the reflected wave will always be inverted.

6.3 Energy Along a Vibrating String

We have some piece of the string with mass dm, and we have some change in the arclength, which is essentially a line ds, which can be broken up into some change in x and some change in y, which gives us that

$$ds^2 = (dx^2 + dy^2)$$

We can see that $dm = \rho\, dx$ (approximating that $ds \approx dx$), and we can rewrite ds^2:

$$ds = dx(1 + (\frac{dy}{dx})^2)^{\frac{1}{2}} \approx dx(1 + \frac{1}{2}(\frac{dy}{dx})^2)$$

Where we've done a Taylor expansion, under the assumption that ds is small.

We want to find the kinetic energy of the system. In this case:

$$E = \frac{1}{2} dm\, v_y^2$$

Where v_y is the transverse velocity (the oscillations going up and down, not the velocity of the propagation of the wave). We can write that

$$v_y = \frac{\partial y(x, t)}{\partial t}$$

Leaving us with

$$E_k = \frac{1}{2}\rho\, dx(\frac{\partial y(x, t)}{\partial t})^2$$

If we do kinetic energy per unit length:

$$E_k = \frac{1}{2}\rho(\frac{\partial y}{\partial t})^2$$

Looking now at the potential energy in the system, we know that potential energy is the work done against the force T:

$$\int_{dx}^{ds} T\, ds$$

Note that we assume that the potential energy is 0 when the string is flat.

$$\begin{aligned}\int_{dx}^{ds} T\, ds &= T(ds - dx) \\ &= T(dx(1 + \frac{1}{2}(\frac{\partial y}{\partial x})^2) - dx) \\ &= T\, dx(\frac{1}{2}(\frac{\partial y}{\partial x})^2)\end{aligned}$$

The potential energy per unit length becomes

$$\frac{1}{2}T(\frac{\partial y}{\partial x})^2$$

We have now found expressions for both E_k and E_p. If we take our solution

$$y = f(ct - x)$$

we see that $\frac{\partial y}{\partial t} = cf'$, and we see that $\frac{\partial y}{\partial x} = -f'$, giving us the relationship that

$$\boxed{\frac{\partial y}{\partial t} = -c\frac{\partial y}{\partial x}} \rightarrow c^2 = \boxed{\frac{T}{\rho}}$$

We can take this relationship and use it in the expression for kinetic and potential energies, and we see that

$$E_k(x,t) = \frac{1}{2}\rho(\frac{\partial y}{\partial t})^2 \qquad E_p(x,t) = \frac{1}{2}\rho(\frac{\partial y}{\partial t})^2$$

We know that the total energy per unit length is the sum of these two:

$$E_T(x,t) = \rho(\frac{\partial y}{\partial t})^2$$

If we compute the derivatives using $y = A\sin(\omega t - kx)$:

$$E_T = \rho\omega^2 A^2 \cos^2(\omega t - kx)$$

If we now take the average of the total energy per unit length over one period:

$$\begin{aligned} E_{Avg} &= \frac{1}{T}\int E_T(x,t)\,dt \\ &= \frac{1}{2}\rho\omega^2 A^2 \end{aligned}$$

This looks familiar, as this is what we get from a simple harmonic oscillator.

Let's now look at the rate of energy flow (averaged over a period), which is equal to the energy at any point, times the velocity. We claim that power is the energy delivered (to maintain the wave) per second. We can try to find out how far the wave travels in 1 second, which we know to be c, so the power is equal to the energy per unit length times c.

Looking at the initial wave:

$$\rho_2\omega^2 A_1^2 c_1$$

And looking at the reflected wave:

$$\rho_1\omega^2 B^2 c_1$$

And finally looking at the transmitted wave:

$$\rho_2\omega^2 A_2^2 c_2$$

If we want to find the ratio of reflected power to initial power:

$$\frac{B^2}{A^2} = (\frac{k_1 - k_2}{k_1 + k_2})^2$$

And we can find the ratio between the transmitted power and the initial power:

$$\frac{\rho_2\omega^2 A_2^2 c_2}{\rho_1\omega^2 A_1^2 c_1} = \frac{\rho_2 c_2}{\rho_1 c_1}(\frac{A_2}{A_1})^2 = \frac{\rho_2 c_2}{\rho_1 c_1}(\frac{2k_1}{k_1 + k_2})^2$$

The quantity ρc is known as the characteristic impedance, and is denoted by Z:

$$Z = \rho c = \frac{T}{c} = (\frac{T}{\omega})k$$

Using this to rewrite that last ratio:

$$\frac{\frac{Tk_2}{\omega}}{\frac{Tk_1}{\omega}}(\frac{2k_1}{k_1+k_2})^2 = \frac{4k_1k_2}{(k_1+k_2)^2}$$

And if we take this new ratio and and it to the previous one (transmitted/initial + reflected/initial), we see that we get 1. We see that the overall energy flow is conserved, which is what we expected.

6.4 Standing Waves

If we have a string of length l, pinned at both ends by walls, and we have some initial travelling wave (first term), we have some of it being reflected (second term).

$$y = ae^{i(\omega t - kx)} + be^{i(\omega t + kx)}$$

If we assume that at $x = 0$, $y = 0$, we can insert the boundary conditions, and we see that

$$y(0,0) = a + b = 0$$

Which tells us that $a = -b$. Inserting this fact into the equation for y:

$$\begin{aligned} y &= a(e^{i(\omega t - kx)} - e^{i(\omega t + kx)}) \\ &= ae^{i\omega t}(e^{-ikx} - e^{ikx}) \\ &= -2aie^{i\omega t}\sin(kx) \end{aligned}$$

Taking the constants and putting them all into one constant A:

$$y = Ae^{i\omega t}\sin(kx)$$

We can now look at $x = l$, where we know that $y = 0$. Looking at $t = 0$, $x = l$:

$$y(l,0) = A\sin(kl) = 0$$

Which tells us that $kl = n\pi$, so $\frac{2\pi}{\lambda}l = n\pi$m and thus

$$\boxed{\lambda_n = \frac{2l}{n}}$$

And from there we can get the frequency:

$$\boxed{\omega_n = \frac{n\pi c}{l}}$$

These are the normal modes, also known as the harmonics of oscillations. This means that the lowest frequency allowed is ω_1:

$$\omega_1 = \frac{c\pi}{l}$$

And the longest wavelength allowed is given by λ_1.

What happens if one of the ends is free? If we have the right end be a massless ring that is allowed to move up and down, we have the same initial moving wave term, but what happens at the right end. There can't be a transverse force, which means that

$$-T\frac{\partial y}{\partial x} = 0 \to Ae^{i\omega t}k\cos(kl) = 0 \to kl = (2n+1)\frac{\pi}{2}$$

We can see that the frequencies that are allowed are different.

6.5 Travelling Waves

Let's do an example of a travelling wave. Let's say we have a very long string (100 meters). If we have a wave given by $y = A\sin(10t - 5x)$, which is right moving. We can ask what the frequency is, which we can see is 10 rad/s, and this gives us a $\nu = \frac{5}{\pi}$. We can ask for the speed of the wave as it travels, which we can find, and we can also ask about the tension, which we can get from $c = \sqrt{\frac{T}{\rho}}$. To find the transverse velocity of the wave, we can just take the derivative with respect to t of y. From this, we have all the parameters of this travelling wave.

We can also ask about the first harmonic frequency (also called the fundamental frequency) of the wave when it hits the other end and turns into a standing wave. The first harmonic of a standing wave occurs when $\lambda = 2L$, and $\lambda = \frac{c}{\nu}$, giving us

$$\nu_1 = \frac{c}{2L} \qquad \nu_n = n\frac{C}{2L}$$

There are two types of velocities when looking at a travelling wave. One of them is the speed/velocity of the the wave being transmitted, and the other is the transverse velocity. This first velocity is known as the phase velocity, and is denote by c or v. Essentially, the phase velocity is the speed by which a constant phase is travelling. This can be seen if we track a single point, such as a crest of a wave. If we have the wave

$$y = A\sin(\omega t - kx)$$

We can define the phase velocity $\phi(x,t)$, which we know is $\sqrt{\frac{T}{\rho}}$.

$$\phi(x,t) = \omega t - kx$$

$$\frac{dx}{dt} = \frac{\omega}{k} = c$$

Now consider two travelling waves on a string:

$$y_1 = a\cos(\omega_1 t - k_1 x) \qquad y_2 = a\cos(\omega_2 t - k_2 x)$$

The total displacement at x and t is equal to the sum:

$$y = y_1 + y_2$$

which is a superposition.

$$y = a(\cos(\omega_1 t - k_1 x) + \cos(\omega_2 t - k_2 x))$$

$$= 2a\cos(\frac{(\omega_1 - \omega_2)}{2}t - \frac{(k_1 - k_2)}{2}x)\cos(\frac{(\omega_1 + \omega_2)}{2}t - \frac{(k_1 + k_2)}{2}x)$$

We saw something similar to this in SHM when talking about beats, and we see that we have similar enveloping oscillations, with the rapid frequency oscillation ($\frac{\omega_1 - \omega_2}{2}$) being enveloped by a slower oscillation frequency (($\frac{\omega_1+\omega_2}{2}$)). The speed of the fast oscillation (also known as the phase velocity) can be given by

$$v_f = \frac{\frac{\omega_1+\omega_2}{2}}{\frac{k_1+k_2}{2}} = \frac{\omega_1 + \omega_2}{k_1 + k_2}$$

The slow velocity, know as v_g (g stands for group):

$$v_g = \frac{\omega_1 - \omega_2}{k_1 - k_2}$$

We can ask ourselves whether these two are the same, i.e, is $v_g = v_p$? We know that $v_p = \sqrt{\frac{T}{\rho}} = \frac{\omega_1}{k_1} = \frac{\omega_2}{k_2}$, and via this we can easily show that $v_g = v_p$.

But what would happen if v_p was a function of ω ($v_p(\omega)$)? (This is known as a dispersive medium) We see that in that case, v_g is not necessarily equal to v_p. If we look at v_g in the limit where ω_1 is close to ω_2:

$$v_g = \frac{d\omega}{dk}$$

For standing waves, the normal modes can be given by $\nu_j = j\nu_1$. However, for a loaded string, we previously found that $\omega_j = 2\omega_0 \sin(\frac{j\pi}{2(n+1)})$. We see that the normal modes here are not similar at all! Can we go from the loaded string to the string? If we take very large m and small j, we can simplify $\sin(\frac{j\pi}{2(n+1)}) \approx \frac{j\pi}{2(n+1)}$, leaving us with

$$\omega_j = 2\omega_0 \frac{j\pi}{2(n+1)}$$

Remember that

$$\omega_0 = (\frac{T}{ma})^{\frac{1}{2}} = \frac{1}{a}(\frac{T}{\frac{m}{a}})^{\frac{1}{2}}$$

$$\omega_0 = \frac{1}{a}(\frac{T}{\rho})^{\frac{1}{2}}$$

Inserting these into ω_j:

$$\omega_j = \frac{2}{a}(\frac{T}{\rho})^{\frac{1}{2}}(\frac{j\pi}{2(n+1)})$$

Note that $a(n+1) = L$.

$$\omega_j = \frac{c}{L}j\pi$$

$$2\pi\nu_j = \frac{c}{L}j\pi$$

$$\nu_j = \frac{c}{2L}j$$

This is exactly the result we got from the strings (tied up at both ends). However, this is only when j is small and n is very large, but what about the case where j is close to n. If we do this case out, we see that the velocity depends on the frequency. Also a general fact, higher normal modes travel slower.

If we go back to a loaded string with masses at distances a apart from each other, we have that

$$m\frac{\partial^2 y_r}{\partial t^2} = \frac{T}{a}(y_{r-1} - 2y_r + y_{r+1})$$

What happens if we have a travelling wave on this:

$$y_r = ae^{i(\omega t - kx)}$$

In this case, x is discrete, so we have to label it $x = ra$:

$$y_r = ae^{i(\omega t - kra)}$$

If we plug this into the first equation:

$$-m\omega^2 e^{i(\omega t - kra)} = \frac{T}{a}e^{i\omega t}(e^{-ik(r-1)a} - 2e^{ikra} + e^{-ik(r+1)a})$$

Factoring out $e^{i\omega t}$ and e^{ikra}:

$$-m\omega^2 = \frac{T}{a}(e^{ika} + e^{-ika} - 2)$$

We can rewrite this as

$$-m\omega^2 = \frac{T}{a}(e^{\frac{ika}{2}} - e^{-\frac{ika}{2}})^2$$

We can then simplify this down to

$$m\omega^2 = \frac{T}{a}(2i\sin(\frac{ka}{2}))^2$$

$$-m\omega^2 = \frac{-4T}{a}\sin^2(\frac{ka}{2})$$

Solving this for ω^2:

$$\omega^2 = \frac{4T}{am}\sin^2(\frac{ka}{2})$$

For a travelling wave in a non-dispersive medium, we would have expected $\omega/k = c$, but we see that that is not true. We also see that the maximum frequency is $\sqrt{\frac{4T}{am}}$, the cutoff frequency. We see that for long wavelengths (same as low frequency), ka is small (via the relationship $k = \frac{2\pi}{\lambda}$). Using the small angle approximation:

$$\sin\frac{ka}{2} \approx \frac{ka}{2}$$

We see that

$$\omega^2 = \frac{4T}{ma}\frac{k^2a^2}{4} = \frac{Tk^2a}{m}$$

$$\frac{\omega^2}{k^2} = \frac{T}{\frac{m}{a}} = \frac{T}{\rho} = c^2$$

We see that this is the behavior of a non dispersive medium. This means that for low frequencies, the medium is non-dispersive.

What if we have a superposition of a group of waves? If we have waves of frequencies $\omega, \omega + \delta\omega, \omega + 2\delta\omega, \ldots, \omega + n\delta\omega$. What would the total wave look like? Each of the waves can take the form:

$$y_0(x,t) = a\cos(\omega t - kx) \qquad y_1(x,t) = a\cos((\omega + \delta\omega)t - (k + \delta k)x)$$

For a moment, lets let $x = 0$. If we do this, and take the sum of all of these waves:

$$R = \sum_{j=0}^{n-1} a\cos((\omega + j\delta\omega)t)$$

We can take the real part of this:

$$R = Re[\sum_{j=0}^{n-1} ae^{i(\omega+j\delta\omega)t}]$$

We call the argument to the Re function Z:

$$Z = \sum_{j=0}^{n-1} ae^{i(\omega+j\delta\omega)t}$$

We can multiply both sides by $e^{i\delta\omega t}$:

$$e^{i\delta\omega t}Z = \sum_{j=0}^{n-1} ae^{i(\omega+(j+1)\delta\omega)t}$$

If we take this equation, and subtract the original equation from it:

$$Z(e^{i\delta\omega t} - 1) = a(e^{i(\omega+n\delta\omega)t} - e^{i\omega t})$$

Leaving us with

$$Z = a\frac{e^{i(\omega+n\delta\omega)t} - e^{i\omega t}}{e^{i\delta\omega t} - 1}$$

Computing the real part of this leaves us with

$$R = a\frac{\sin(\frac{n\delta\omega t}{2})}{\sin(\frac{\delta\omega t}{2})}\cos(\bar{\omega}t)$$

Where $\bar{\omega} = \omega + \frac{1}{2}(n-1)\delta\omega$. We see that we have a high frequency oscillation $\omega = \bar{\omega}$, and a slow frequency envelope.

We can do something else interesting with this equation. If we call $n\delta\omega = \Delta\omega$, then

$$R = a\frac{\sin(\frac{\Delta\omega t}{2})}{\sin(\frac{\Delta\omega t}{2n})}\cos(\bar{\omega}t)$$

For very large n, then $\frac{\Delta\omega}{2n} \to 0$, so we can use a small angle approximation:

$$R = an\frac{\sin\frac{\Delta\omega t}{2}}{\frac{\Delta\omega t}{2}}\cos(\bar{\omega}t)$$

If we look at when this is 0, we need $\frac{\Delta\omega t}{2} = \pi$, which means that when $t = \frac{2\pi}{\Delta\omega}$, we have a 0. This is essential the width of the time profile of this pulse, Δt:

$$\Delta t = \frac{2\pi}{\Delta\omega}$$

This is the half width of the time profile. Using this relationship, and the relationship $\omega = 2\pi\nu$, we can see that

$$\Delta t \Delta \nu = 1$$

This is important, because it gives us a relationship between ν and t.

Take for example an atomic transition with a lifetime of τ. We can get that the frequency has a width of $\Delta\nu$:

$$\tau \Delta \nu \approx 1$$

Remember that from Planck and Einstein:

$$E = h\nu \to \Delta\nu = \frac{\Delta E}{h}$$

$$\Delta t \frac{\Delta E}{h} = 1 \to \Delta t \Delta E \approx h$$

This is the Heisenberg Uncertainty Principle.

7 Fourier Series

If we have a reasonably behaved periodic function (can have discontinuities, but they have to be reasonable.), we can describe it as a sum of sinusoidal waves. For example, we can approximate a square wave of frequency 5 Hz and amplitude 1 as a superposition of n sinusoids. One caveat is that we only want n to be an odd number, which we will talk about later. If we look at $n = 199$, it looks almost perfect, so if we take $n \to \infty$, we can in theory get a perfect approximation.

This is a practical intro to Fourier series. Fourier's theorem tells us that we can write a periodic functions as a sum of cosine and sine functions. This is very powerful, and it has many different applications.

The question is, how do we know which sinusoids to add? And how do we know whether or not we should be using sines or cosines?

Theorem 7.1 (Fourier Series Theorem)**.** *Any reasonably well behaved periodic function (including piecewise discontinuities), can be represented by:*

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(nx) + b_n \sin(nx)]$$

We want to find out what a_n and b_n are. To do this, we have to talk about **orthogonal** functions:

$$\int_0^{2\pi} \sin(nx)\sin(mx)\,dx = 0,\ n \neq m$$

$$\int_0^{2\pi} \cos(nx)\cos(mx)\,dx = 0,\ n \neq m$$

$$\int_0^{2\pi} \sin(nx)\cos(mx)\,dx = 0$$

We can prove these statements by just doing the integrals out, and we see that if $n = m$, we have π, and if $n \neq m$, we get 0.

Looking back at $f(x)$, we can multiply the function by $\cos(mx)$ and integrate:

$$\int_0^{2\pi} f(x)\cos(mx)\,dx =$$

$$= \sum_{n=1}^{\infty} a_n \int_0^{2\pi} \cos(nx)\cos(mx)\,dx + \sum_{n=1}^{\infty} b_n \int_0^{2pi} \sin(nx)\cos(mx)\,dx$$

From what we just learned via orthogonal functions, we can see that the second term will always be 0, regardless of m and n. The first term is 0 for $n \neq m$, which gives us

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos(mx)\,dx$$

If we multiply $f(x)$ by $\sin(mx)$ and integrate, we see that via orthogonality we have a similar situation, leaving us with

$$\int_0^{2\pi} f(x)\sin(mx)\,dx = \pi b_n$$

Putting all of this together:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}[a_n\cos(nx) + b_n\sin(nx)]$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos(nx)\,dx$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin(nx)\,dx$$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$$

Notice that if $f(x)$ is an even function, then $b_n = 0$, and if $f(x)$ is odd, we have that $a_n = 0$, which means that we only need one of the trig functions.

Let's try to apply this to a square wave. We can split the integrals about the discontinuities:

$$a_n = \frac{1}{\pi}\int_0^{\pi} \cos(mx)\,dx - \int_{\pi}^{2\pi} \cos(mx)\,dx$$

$$= \frac{1}{\pi}[(\frac{1}{m}\sin(mx))_0^{\pi} - (\frac{1}{m}\sin(mx))_{\pi}^{2\pi}] = 0$$

Doing the b terms:

$$\begin{aligned} b_n &= \frac{1}{\pi}[\int_0^{\pi} \sin(mx)\,dx - \int_{\pi}^{2\pi} \sin(mx)\,dx] \\ &= \frac{1}{\pi}\frac{4}{n} \\ &= \frac{4}{n\pi} \end{aligned}$$

We can also see that $a_0 = 0$. We can now put this all together:

$$a_0 = 0 \qquad a_n = 0 \qquad b_n = \frac{4}{n\pi}$$

where n is odd.

$$f(x) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{-1}{n}\sin(nx) \;\; n = \text{odd}$$

we can also write this as

$$f(x)\frac{4}{\pi}\sum_{n=0}^{\infty}\frac{-1}{2n+1}\sin((2n+1)x)$$

Let's look at this. If we let $x = \frac{\pi}{2}$, we see that $f(x) = 1$, this tells us that 1 is equal to the series that we can generate from f:

$$1 = \frac{4}{\pi}(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots)$$

What happens exactly at the discontinuities? We see that our function just represents the average of the two discontinuity values, which is 0:

$$Series(x) = \frac{f(x^+) + f(x^-)}{2} = 0$$

The reality is that very close to the discontinuity, the series will exceed/overshoot the value of the wave by about 9 percent of the jump, no matter how many frequencies we add in. This is known as the Gibb's phenomenon.

8 Longitudinal Waves

So far we have talked about transverse waves, but what about other types of waves? What about deformation in solids, liquids, and gases? If we have a rod, and we apply a force F at one end of it. We know that the rod has some elasticity, given by the Young's modulus:

$$E = \frac{\frac{F}{A}}{-\frac{\Delta L}{L}}$$

Where A is the cross-sectional area, and the denominator is the change in the length. The values of the modulus can be very large, depending on the material, due to different atom bindings and structure. For aluminum, $E = 69$ GPa (gigapascals, a unit of pressure), and for glass $E = 50 - 90$ GPa. If E is a very large number, then the change in the length must be very small. This is why solids seem very nondeformable, although they are. If we think about sound waves travelling through the solid, we can intuitively guess that the wave velocity c will have to do with the density and the Young's modulus:

$$c \propto \sqrt{\frac{E}{\rho}}$$

These waves will be longitudinal, as we can think of the solid as an infinite number of coupled springs, which when moved form longitudinal waves. The direction of the propogation of the deformation is in the same direction as the deformation, whereas in a transverse wave, the propogation was along the string and the deformation was up and down.

8.1 Waves in Gases

What about gases? Gases are in general characterized by pressure P, volume V, and temperature T. For ideal gases, we have the well known $PV = nRT$. If we apply a force to a column of gas, we're adding some extra pressure:

$$P = \frac{F}{A}$$

The Bulk Modulus B_a is the analog to the Young's modulus:

$$B_a = -\frac{\Delta P}{\frac{\Delta V}{V}}$$

For air, $B_a = 140$ kPa.

What happens when a gas is disturbed? We can talk about an adiabatic process, which is when no heat is exchanged with the outside of the system. This is an approximate assumption, because we can assume that the disturbance is very fast. We have that for an adiabatic process:

$$PV^\gamma = c$$

where $\gamma = \frac{C_P}{C_v}$, or the specific heat at a constant pressure and a specific heat at a constant volume. We can then take a differential of this:

$$\gamma PV^{\gamma-1}\Delta V + V^\gamma \Delta P = 0$$

Simplifying:

$$\boxed{B_a = \gamma P_0}$$

Let's start with a column of gas with cross-sectional area $A = 1$. We have some initial pressure P_0, and some initial volume $\Delta xA = \Delta x$. We have some variable excess pressure being added via the function $P(x, t)$. After the disturbance, the net force is given by

$$F = A(P'(x, t) - P'(x + \Delta x, t)) = P'(x, t) - P'(x + \Delta x, t) = -\frac{dP'}{dx}\Delta x$$

Using Newton's Law:

$$-\frac{dP'}{dx}\Delta x = \rho \Delta x \frac{d^2\eta}{dt^2}$$

$$-\frac{dP'}{dx} = \rho \frac{d^2\eta}{dt^2}$$

We have that $B = \frac{-\Delta P}{\frac{\Delta V}{V}}$:

$$\Delta V = \eta(x + \Delta x, t) - \eta(x) = \frac{d\eta}{dx}\Delta x$$

This gives us that

$$\frac{\Delta V}{V} = \frac{d\eta}{dx}$$

$$\boxed{P' = -B\frac{d\eta}{dx}}$$

We can use this, and we're left with the wave equation:

$$\frac{\partial^2\eta}{\partial x^2} = \frac{\rho}{B_a}\frac{\partial^2\eta}{\partial t^2}$$

This is just the wave equation, and tells us that $c = \sqrt{\frac{B_a}{\rho}}$. We have seen that $c = \sqrt{\frac{B_a}{\rho}}$ for a longitudinal wave in a gas. Is this a constant? We have that $B_a = \gamma P_0$, and for a diatomic gas like air, this is $\frac{7}{5}$. This gives us that $C = \sqrt{\frac{\gamma P_0}{\rho}}$. We see that this depends on the initial pressure. If we take air for example, with $P_{air} = 1.3$ kilograms per cubic meter, at 1 atmosphere P_0, which is roughly 10^5 Pa. This tells us that $c \approx 330$ m/s, which tells us that our model is pretty good. The assumption that we made was that the process was adiabatic, adding in the factor γ. This is a good assumption because we're saying that the disturbance is happening very fast, so the tube of gas isn't able to exchange heat with the outside.

If we use the ideal gas law:

$$PV = nRT$$

If we insert this into c:

$$c = \sqrt{\frac{\gamma(nRT)}{\rho V}} = \sqrt{\frac{\gamma RT}{m}}$$

where m is the molar mass. We can see that the velocity of sound depends on temperature of the gas. There is another thing we can do to the expression. If we again use the ideal gas law, written as $PV = NkT$, where N is the number of molecules and k is the Boltzmann constant. This gives us

$$c = \sqrt{\frac{\gamma NkT}{\rho V}} = \sqrt{\frac{\gamma kT}{\frac{\rho V}{N}}} = \sqrt{\frac{\gamma kT}{m}}$$

where m is the molecular mass. If we now square:

$$c^2 = \gamma\frac{kT}{m}$$

If we think about thermo, if we think about the 3 degrees of freedom translationally(x, y, z), each one has energy $\frac{1}{2}kT$, so the kinetic energy is given as $\frac{3}{2}kT$. We can then set this equal to the mean kinetic energy:

$$\frac{3}{2}kT = \frac{1}{2}m\bar{v}_{rms}^2$$

If we solve for kT:

$$kT = \frac{m\bar{v}_{rms}^2}{3}$$

If we put these together, we have that

$$c \approx .7\bar{v}_{rms}$$

Let's move back to the wave equation

$$\frac{\partial^2\eta}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2\eta}{\partial y^2}$$

The solution to this will be the usual

$$\eta = \eta_0 e^{i(\omega t - kx)}$$

for right moving waves, and

$$\eta = \eta_0 e^{i(\omega t + kx)}$$

for left moving waves. We also want to keep in mind the relationship between the excess pressure and the wave:

$$P' = -B_a \frac{\partial \eta}{\partial x} = iB_a k\eta$$

For right moving, and is equal to $-iB_a k\eta$ for left moving waves. Note that η and P' are 90 degrees out of phase due to the i that is present in their relationship.

8.1.1 Energy in the Wave

We can compute the change in kinetic energy in the system ΔE_{kin} (per unit volume):

$$\Delta E_{kin} = \frac{1}{2}\rho\dot{\eta}^2$$

We have that $\eta = \eta_0 e^{i(\omega t - kx)}$. If we compute $\dot{\eta}$:

$$\dot{\eta} = i\omega\eta$$

Taking the real part of this:

$$Re(\dot{\eta}) = -\omega\eta_0 \sin(\omega t - kx)$$

This gives us that

$$\dot{\eta}^2 = \omega^2\eta_0^2 \sin^2(\omega t - kx)$$

If we now average $\dot{\eta}^2$ over many wavelengths:

$$\bar{\dot{\eta}}^2 = \frac{\omega^2\eta_0^2}{n\lambda} \int_0^{n\lambda} \sin^2(\frac{2\pi}{\lambda}(ct - x))\,dx$$

This gives us that

$$\Delta E_{kin} = \frac{1}{2}\rho\bar{\dot{\eta}}^2$$

and that

$$\Delta \bar{E}_{kin} = \frac{1}{4}\rho\omega^2\eta_0^2$$

We will assume that $\Delta\bar{E}_{pot} = \Delta\bar{E}_{kin}$, and as such:

$$\Delta E_{total} = \frac{1}{2}\rho\omega^2\eta_0^2$$

What about the energy flow? Energy flux is $\bar{E}_{total} \times c$. This tells us that the flux is:

$$\Phi = \frac{1}{2}\rho\omega^2\eta_0^2 c$$

For sound waves, this is from about 10^{-12} Watts per square meter to 1 (Threshold of hearing, and where things get painful). This is known as the intensity, I, and the threshold is known as I_0. Generally we use the logarithm scale of decibels, where we convert via $10\log_{10}\frac{I}{I_0}$.

8.1.2 Boundary Conditions

If one end of the tube is closed, let's say the left end, we have that $\eta(x = 0) = 0$. If on the other hand it is open, we have that $P' = 0$ at $x = 0$, which tells us that

$$-B_a \frac{\partial \eta}{\partial x} = 0 \to \frac{\partial \eta}{\partial x} = 0$$

8.1.3 Doppler Effect

We have some wave source S, which is putting out waves in all directions. If we have an observer some distance away from the source, and either the source begins to move or the observer begins to move, we want to see what will happen. We know that wavelength if given by $cT = \frac{c}{\nu}$. The wavelength corresponds to the distance between two wavefronts/crests, or we can think of it via the frequency (number of wavelengths in the distance travelled in 1 second).

The Doppler effect occurs when the wavelength changes as the source moves with some velocity v_s:

$$\lambda' = cT - v_s T$$

$$\frac{c}{\nu'} = \frac{c - v_s}{\nu}$$

For frequencies:

$$\nu' = \frac{c}{c - v_s}$$

In general, the Doppler effect states that

$$\nu' = \frac{c}{c \pm v_s}$$

If the observer is moving towards the source with some speed v_o, we can just use relative velocities and change our frame of reference, and we'll see that

$$\nu' = \frac{c + v_o}{c} \nu$$

If we generalize this to all cases, with both source and observer moving:

$$\boxed{\nu' = \nu \frac{c \mp v_o}{c \pm v_s}}$$

We are assuming that c is the same in all cases. This tells us that frequency changes by motion of the source and the observer, which is very important.

Let's say we have a distant star, and we want to see whether the star is moving away from us/toward us, and the speed at which it is moving. How can we know whether it is moving? Since stars are mostly made of hydrogen and helium. This gives the star a spectrum, which we know the wavelengths of. What we can do is take the hydrogen wavelength and notice that the pattern in the star is shifted, either up or down in wavelength. If the wavelength is shorter, it is blueshifted, and if it is shifted up, it is redshifted. Blueshifted stars are moving towards us, and redshifted stars are moving away from us. Once we know how much they are shifted by, we can compute the velocity of the star.

Using this, we learned that the further away a star/galaxy is from Earth, the faster it moves away from us, giving us the relationship $v = Hr$, where v is the velocity, H is Hubble's constant, and r is the distance from Earth. This showed that the universe was expanding.

We can also compute how hot the star is with the Doppler effect. If we have molecules in a gas, temperature is a measure of the random motion of the molecules, and we characterize that as $\frac{1}{2}kT$ times the number of degrees of freedom. This is equal to the kinetic energy:

$$\frac{1}{2}m\bar{v}^2 = \frac{3}{2}kT$$

We can then get the average velocity squared:

$$\bar{v}^2 = \frac{3kT}{m}$$

Because sometimes the molecules are moving in random directions, sometimes the wavelength is shifted either up or down. If we look at the width of the wavelength distribution, we can relate that to the average velocity via the Doppler effect.

If we have a moving source where $v_s > c$, we can see that we'll have a cone created, where $\sin\theta = \frac{ct}{v_s t} = \frac{c}{v_s}$, where θ is the half angle of the cone. This is like what happens with fighter jets breaking the sound barrier, where the cone is actually physicall visible.

9 Transmission Lines

Say we have two conductors separated by some nonconducting medium (dielectric). One common example of this is a coaxial cable, which is just a cylinder which has a central conductor of radius r_1, and an external conductor with a radius r_2. In between these conductors is insulating material. We can put some voltage source in the system (a battery for example), and some sort of load, like a resistor or a lightbulb. We can then put a switch that completes the circuit. If we close the switch at $t = 0$, a voltage is applied across the conductor. How long will it take for the lightbulb to 'sense' that there is some voltage?

We know that when we create a current in the conductor, we have a magnetic field being formed. We then have some magnetic flux being generated by the B field by the gap between the two conductors. We can think of this as an inductor:

$$\Phi_B = LI$$

Where L is the inductance, and is measured in Henry's. If we plot the voltage as a factor of time, we see we have a change from 0 to some V_0, giving us some ΔV. This means that we have some ΔI, and by Faraday's Law:

$$\epsilon = -\frac{d\Phi_B}{dt} = -L\frac{dI}{dt} = \Delta V$$

Note that this is in the case of a switch, where the voltage is instantly generated. If we have some generator, we would have some sinusoidal generation of voltage. However, our work with Fourier series tells us that we can model a step function with a sum of sinusoids.

On the other hand we can think of this as a capacitor, which has some electric field being generated:

$$q = CV$$

$$\frac{dq}{dt} = C\frac{dV}{dt} = \Delta I$$

For a coaxial cable:

$$L_0 = \frac{\mu}{2\pi} \ln\left(\frac{r_2}{r_1}\right)$$

$$C_0 = \frac{2\pi\epsilon}{\ln\left(\frac{r_2}{r_1}\right)}$$

Where L_0 is the inductance per meter, and C_0 is the capacitance per meter.

9.1 Wave Equation

Lets take our two conductors and break them up into small sections. Each section represents some inductance $\Delta L = L_0 \Delta x$, and a capacitance $\Delta C = C_0 \Delta x$. Note that we are avoiding the inherent resistances in the conductors, and assuming the insulator between the two is a perfect insulator. We will ignore this for now, but we'll come back to it. Remember that an LC circuit is an oscillator, so we're coupling a bunch of oscillators together. We can see the parallels to coupled springs, and we will end up with a wave equation that describes this system.

Across a given inductor, when we have a change in the current, we have a change in the voltage ΔV. We can label each inductor by the position, the first being $V(x,t)$, and the second being $V(x+\Delta x, t)$, and so on. We have some change in voltage in the section:

$$\Delta V = V(x+\Delta x, t) - V(x,t) = \frac{dV}{dx}\Delta x$$

We also know that

$$\Delta V = -(L_0 \Delta x)\frac{dI}{dt}$$

Combining these two equations, we have

$$\boxed{\frac{dV}{dx} = -L_0 \frac{dI}{dt}}$$

Let's try and connect the currents. We have a capacitor with a change in the current, taking some of the current, giving us some $-\Delta I$ (negative since the capacitor is taking the current). We end up having

$$\Delta I = I(x+\Delta x, t) - I(x,t) = \frac{dI}{dx}\Delta x$$

And we have that

$$-\Delta I = \frac{dq}{dt} = C_0 \Delta x \frac{dV}{dt}$$

Putting these two together:

$$\boxed{-\frac{dI}{dx} = C_0 \frac{dV}{dt}}$$

We now have two equations for the system, one given via inductance and the other given via the capacitance.

If we take a derivative with respect to x of $\frac{dV}{dx}$:

$$\frac{d}{dx}\frac{dV}{dx} = -L_0 \frac{d^2 I}{dx\, dt}$$

Which gives us

$$\frac{d^2V}{dx^2} = -L_0\frac{d^2I}{dx\,dt}$$

We can instead take the derivative with respect to time of the first equation, giving us

$$-\frac{d^2I}{dxdt} = C_0\frac{d^2V}{dt^2}$$

We can insert this into the second equation:

$$\boxed{\frac{d^2V}{dx^2} = C_0L_0\frac{d^2V}{dt^2}}$$

This is in the same form as the wave equation, giving us that $c^2 = \frac{1}{C_0L_0}$:

$$\frac{d^2V}{dx^2} = \frac{1}{c^2}\frac{d^2V}{dt^2}$$

This means that if we generate a change in the voltage at one point in the transmission line, the change in the voltage will set off an electric field and a magnetic field, which will travel as if they were waves, and the speed of this wave is given by $\sqrt{\frac{1}{L_0C_0}}$.

Lets look at a coaxial cable. We have equations for L_0 and C_0:

$$L_0 = \frac{\mu}{2\pi}\ln\left(\frac{r_2}{r_1}\right)$$

$$C_0 = \frac{2\pi\epsilon}{\ln\left(\frac{r_2}{r_1}\right)}$$

We know that $\mu = \mu_0\mu_r$, and that $\epsilon = \epsilon_0\epsilon_r$, where μ_r and ϵ_r are material dependent. From these, we get that

$$c^2 = \frac{1}{L_0C_0} = \frac{1}{\mu_0\epsilon_0(\epsilon_r\mu_r)}$$

Giving us that

$$c = \frac{1}{\sqrt{\mu_0\epsilon_0}}$$

In air, the speed is about the speed of light.

9.2 Impedance

We now want to find what the impedance of the transmission line is. We define it to be

$$Z = \frac{V(x,t)}{I(x,t)}$$

This has units of Ohms. Why do we call this impedance rather than resistance? We do this because this ratio isn't that simple, because it has an amplitude and phase, because V and I are complex valued. Before we can see what values these take, we need to find what the solutions of this system are.

$$\frac{d^2V}{dx^2} = \frac{1}{c^2}\frac{d^2V}{dt^2}$$

We can write the solutions in the form

$$V_+ = V_0 e^{i(\omega t - kx)}$$

We also have the other equation:

$$I_+ = I_0 e^{i(\omega t - kx)}$$

Note that this is for right moving transmissions/waves (Hence the plus signs). To compute the impedance, we can just use the relationship

$$\frac{dV}{dx} = -L_0 \frac{dI}{dt}$$

and just take the derivatives of our general solutions and plug in:

$$-kV_0 = -L_0 I\omega$$

$$Z_0 = \frac{V_0}{I_0} = \frac{L_0\omega}{k} = L_0 c = \sqrt{\frac{L_0}{C_0}}$$

This value Z_0 is known as the characteristic impedance of the transmission line.

Let's look at a left moving wave, V_-, I_-. The characteristic impedance is given by

$$Z_c = \frac{V_-}{I_-}$$

Using the equation

$$\frac{dV}{dx} = -L_0 \frac{dI}{dt}$$

And the definitions of V_- and I_-:

$$V_- = V_0 e^{i(\omega t + kx)}$$

$$I_- = I_0 e^{i(\omega t + kx)}$$

We end up with $Z_c = -\sqrt{\frac{V_0}{I_0}}$. Note that this value is negative. Now lets look at the characteristic impedance of a coaxial cable. Remember that we have a center conductor of radius r_1, and an outer conductor of radius r_2. Using the expressions we had for L_0 and C_0 for a coaxial cable, and assuming that $\mu = \mu_0$ and $\epsilon = \epsilon_r \epsilon_0$, we end up with

$$|Z_c| = \sqrt{\frac{L_0}{C_0}} = \frac{1}{2\pi}\sqrt{\frac{\mu}{\epsilon}} \ln(\frac{r_2}{r_1})$$

For a typical coaxial cable at home, $|Z_c| = 50\Omega$. To find the velocity of the travelling wave, we use the relationship

$$c = \frac{1}{\sqrt{\mu\epsilon}} = \frac{1}{\sqrt{\mu_0\epsilon_0}}\frac{1}{\sqrt{\epsilon_r}}$$

Let's put a load on the transmission line, like a resistor or a lightbulb. This adds some external load/impedance, Z_L. If we have a wave $V_+ = V_0 e^{i(\omega t - kx)}$, we'll have some reflected wave $V_- = V_0' e^{i(\omega t + kx)}$. Our voltage will be $V = V_+ + V_-$, and our current will be $I = I_+ + I_-$. The ratio between these two, $\frac{V}{I}$, must be equal to Z_L, leading us to a system of equations

$$\frac{V_-}{I_-} = -Z_0 \qquad \frac{V_+}{I_+} = Z_0 \qquad \frac{V_+ + V_-}{I_+ + I_-} = Z_L$$

This gets us that

$$\frac{V_-}{V_+} = \frac{Z_L - Z_0}{Z_L + Z_0}$$

Note that in the case where $Z_L = Z_0$, $V_- = 0$, which means that there is no reflected wave.

The impedance at a different location in a transmission line will depend on x:

$$Z(x) = \frac{V_+(x) + V_-(x)}{I_+(x) + I_-(x)}$$

If the external load is 0, $Z_L = 0$, we see that

$$\frac{V_-}{V_+} = -1$$

Which tells us that $V_- = -V_+$, and $V = V_+ + V_- = V_0 e^{i(\omega t - kx)} - V_0 e^{i(\omega t + kx)}$. This gives us that $V = 2V_0 \sin(kx)\sin(\omega t)$. We can do the same thing with the current, where we notice that $\frac{I_-}{I_+} = 1$, giving us that $I = 2I_0 \cos(kx)\cos(\omega t)$. From this, we can use $P = IV$ to get that

$$P = V_0 I_0 \sin(kx)\cos(kx)\sin(\omega t)\cos(\omega t)$$

Notice that the average power in this expression will be 0.

9.3 Imperfect Conductors/Insulators

So far we have worked with ideal conductors and insulators, but in the real world we have some small resistance R_0 in the conductors, and some conductance G_0 in the insulator, where $G_0 = \frac{1}{R_c}$. This gives us new equations:

$$\frac{dV}{dx} = -L_0 \frac{dI}{dt} - R_0 I$$
$$\frac{dI}{dx} = -C_0 \frac{dV}{dt} - G_0 V$$

These are just the ideal equations with the addition of Ohm's law in the voltage and Kirchoff's Loop laws in the case of the current. We can then do what we did before, giving us a new diffeq:

$$\frac{d^2V}{dx^2} = L_0 C_0 \frac{d^2V}{dt^2} + (L_0 C_0 + R_0 C_0)\frac{dV}{dt} + R_0 G_0 V$$

We can see that this is the ideal wave equation with extra damping terms, and this will have a damped solution:

$$V = Ae^{-\gamma x} e^{i\omega t}$$

or

$$V = Ae^{\gamma x} e^{i\omega t}$$

We know that γ is complex, and so we can split it:

$$\gamma = \alpha + ik$$

which gives us

$$V = Ae^{-\alpha x} e^{i(\omega t - kx)}$$

If we let $R_0 = 0$ and $G_0 = 0$, we expect to see the same results as in the ideal situation. Indeed, if we solve for α, we see that $\alpha = 0$, which drops the exponential term.

9.4 Low Loss Lines

Can we have a low loss transmission line? Looking at the equation:

$$\gamma = [(R_0 + i\omega L_0)(G_0 + i\omega C_0)]^{\frac{1}{2}}$$

Factoring:

$$\gamma = \sqrt{(i\omega L_0)(i\omega C_0)}\left[\left(1 + \frac{R_0}{i\omega L_0}\right)\left(1 + \frac{G_0}{i\omega C_0}\right)\right]^{\frac{1}{2}}$$

For $\frac{R_0}{\omega L_0} \ll 1$, and $\frac{G_0}{\omega C_0} \ll 1$, we can ignore the cross terms, and we are left with

$$i\omega\sqrt{L_0C_0}\left[1 - i\left(\frac{R_0}{2\omega L_0} + \frac{G_0}{2\omega C_0}\right)\right]$$

Where we've done a Taylor expansion. Separating the real and imaginary terms:

$$\gamma = \omega\sqrt{L_0C_0}\left(\frac{R_0}{2\omega L_0} + \frac{G_0}{2\omega C_0}\right) + i\omega\sqrt{L_0C_0}$$

Once again letting $\gamma = \alpha + ik$:

$$\alpha = \sqrt{L_0C_0}\left(\frac{R_0}{2L_0} + \frac{G_0}{2C_0}\right) \qquad k = \omega\sqrt{L_0C_0} = \frac{\omega}{c}$$

It can also be show that the characteristic impedanc eof a lossy transmission line is givne by

$$Z_0 = \sqrt{\frac{R_0 + i\omega L_0}{G_0 + i\omega C_0}} = \frac{V_+}{I_+}$$

Note that this is for a right moving wave, and the characteristice impedance for a left moving wave would be the negative of this.

10 Electromagnetic Waves

10.1 Reminder of Vector Calculus

Remember that we have the ∇ operator:

$$\nabla = \frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z}$$

Remember the definition of the gradient:

$$\nabla f(x,y,z) = \frac{\partial f}{\partial x}\hat{x} + \frac{\partial f}{\partial y}\hat{y} + \frac{\partial f}{\partial z}\hat{z}$$

And the divergence:

$$\nabla \cdot \vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

And finally the curl:

$$\nabla \times A = \left(\frac{dA_z}{dy} - \frac{dA_y}{dz}\right)\hat{x} + \left(\frac{dA_z}{dx} - \frac{dA_x}{dz}\right)\hat{y} + \left(\frac{dA_y}{dx} - \frac{dA_x}{dy}\right)\hat{z}$$

And the Laplacian:

$$\nabla^2 A = \nabla(\nabla \cdot A) - \nabla \times (\nabla \times A)$$

We also have a lot of vector identities that are very useful (like the curl of the curl of a function). We also need to remember some important theorems that connect integrals:

$$\int_a^b (\nabla f)\, dl = f(b) - f(a)$$

$$\int (\nabla \cdot A)\, dV = \oint A \cdot da$$

$$\int (\nabla \times A) \cdot da = \oint A \cdot dl$$

10.2 Maxwell's Equations

Let's start with the Maxwell's equations.

Gauss's Law:

$$\int E \cdot dA = \frac{Q_E}{\epsilon_0} \to \nabla \cdot E = \frac{\rho_E}{\epsilon_0}$$

Gauss's Law for magnetism:

$$\int B \cdot dA = 0 \to \nabla \cdot B = 0$$

Faraday's Law of induction:

$$dE \cdot dl = -\frac{d\Phi_B}{dt} \to \nabla \times E = -\frac{dB}{dt}$$

and Ampere's Law:

$$\int B \cdot dl = \mu_0 \epsilon_0 \frac{d\Phi_E}{dt} + \mu_0 I_E \to \nabla \times B = \mu_0 \epsilon_0 \frac{dE}{dt} + \mu_0 J_E$$

10.3 Deriving the Wave Equation

Let's now use these equations and try to derive the wave equation for EM waves. We want to do these in free space. In free space, $\rho_E = 0$ (no free charges) and $J_E = 0$ (no current density). Maxwell's equations then become

$$\nabla \cdot E = 0 \qquad \nabla \cdot B = 0 \qquad \nabla \times E = -\frac{dB}{dt} \qquad \nabla \times B = \mu_0 \epsilon_0 \frac{dE}{dt}$$

We can take the third equation, and take the curl of it (using a vector identity):

$$\nabla \times (\nabla \times E) = \nabla(\nabla \cdot E) - \nabla^2 E = -\nabla \times (\frac{\partial B}{\partial t})$$

The first term is 0, by Gauss's law:

$$-\nabla^2 E = -\frac{d}{dt}(\nabla \times B) = -\frac{d}{dt}\left(\mu_0 \epsilon_0 \frac{dE}{dt}\right)$$

We are then left with

$$\boxed{\nabla^2 E = \mu_0\epsilon_0 \frac{d^2E}{dt^2}}$$

This looks like a wave equation. We then have that $c^2 = \frac{1}{\epsilon_0\mu_0}$:

$$\nabla^2 E = \frac{1}{c^2}\frac{d^2E}{dt^2}$$

By the definition of the Laplacian:

$$\nabla^2 E_x = \frac{1}{c^2}\frac{d^2E_x}{dt^2} \qquad \nabla^2 E_y = \frac{1}{c^2}\frac{d^2E_y}{dt^2} \qquad \nabla^2 E_z = \frac{1}{c^2}\frac{d^2E_z}{dt^2}$$

We can also show from Maxwell's equations that

$$\nabla^2 B = \frac{1}{c^2}\frac{d^2B}{dt^2}$$

via a similar process.

Looking at the wave equation, we have that $c = \frac{1}{\sqrt{\mu_0\epsilon_0}}$. For a medium where $\epsilon = \epsilon_r\epsilon_0$ and $\mu = \mu_r\mu_0$, we have that

$$c_{medium} = \frac{1}{\sqrt{\mu\epsilon}}$$

and the index of refraction is given by

$$n = \frac{c}{c_{medium}} = \sqrt{\frac{\mu\epsilon}{\mu_0\epsilon_0}}$$

For dielectrics, like a piece of plastic or glass, $\mu \approx \mu_0$, giving us that $n = \frac{\epsilon}{\epsilon_0}$.

10.4 Solution to the Equation

Let's consider E_x:

$$\frac{\partial^2 E_x}{\partial x^2} + \frac{\partial^2 E_x}{\partial y^2} + \frac{\partial^2 E_x}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2 E_x}{\partial t^2}$$

We have a solution $E^x = E_0^x e^{i(\vec{k}\cdot\vec{r}-\omega t)}$, where $\vec{k}\cdot\vec{r} = k_x x + k_y y + k_z z$:

$$E^x = E_0^x e^{i(k_x x + k_y y + k_z z - \omega t)}$$

Note that this is the equation of a right moving wave, and the equation of a left moving wave would have a positive ωt. This same solution works for E^y and E^z, with the same frequency and velocity. If we enter E^x into the wave equation, we'd be left with

$$k_x^2 + k_y^2 + k_z^2 = \frac{\omega^2}{c^2}$$

We see that the left side is the magnitude of $\vec{k}$ squared, and the right side is the square of the wave number, which tells us that we have exactly the same relationship:

$$\frac{\omega^2}{k^2} = c^2$$

And this is the same behavior for E_y and E_z. All of this behavior also applies for B.

So far we have seen that we can have waves of the form

$$E = E_0 e^{i(k\cdot r - \omega t)} \qquad B = B_0 e^{i(k\cdot r - \omega t)}$$

We can learn more from Maxwell's equations. We know that

$$\nabla \cdot E = 0 \qquad \nabla \cdot B = 0 \qquad \nabla \times E = -\frac{dB}{dt} \qquad \nabla \times B = \mu_0 \epsilon_0 \frac{dE}{dt}$$

Looking at the first equation:

$$\nabla \cdot E = 0 \to ik_x E_x + ik_y E_y + ik_z E_z = \vec{k} \cdot E = 0 \to E \perp \vec{k}$$

We see that our electric field is normal to the direction of propagation. We also know that

$$\nabla \cdot B = 0 \to ik_x B_x + ik_y B_y + ik_z B_z = \vec{k} \cdot B = 0 \to \vec{k} \perp B$$

We see that the magnetic field is also normal to the direction of propagation. We have transversely polarized waves. We now want to find the relationship between the electric field and the magnetic field, so we want to use the fact that

$$\nabla \times E = -\frac{\partial B}{\partial t}$$

$$\left(\frac{dE_z}{dy} - \frac{dE_y}{dz}\right)\hat{x} + \left(\frac{dE_z}{dx} - \frac{dE_x}{dz}\right)\hat{y} + \left(\frac{dE_y}{dx} - \frac{dE_x}{dy}\right)\hat{z} = -\frac{dB}{dt}$$

If we do this all out, we have that

$$i(\vec{k} \times E) = -i\omega B \to \vec{k} \times E = \omega B$$

This tells us that $E \perp B$, and $B = \frac{E}{c}$.

10.5 Poynting Vector and Energy Density

We have a Poynting vector $\vec{S}$:

$$\vec{S} = \frac{1}{\mu_0} E \times B$$

This points in the direction of $\vec{k}$, the direction of propagation.

Energy density is given by $\mathcal{E} = \frac{1}{2}\epsilon_0 E^2 + \frac{1}{2}\frac{1}{\mu_0}B^2$. If we take the derivative of this:

$$\frac{\partial \mathcal{E}}{\partial t} = \epsilon_0 E \frac{dE}{dt} + \frac{1}{\mu_0} B \frac{dB}{dt}$$

By Maxwell's Equations:

$$\frac{\partial \mathcal{E}}{dt} = \epsilon_0 E \left(\frac{1}{\mu_0 \epsilon_0} \nabla \times B\right) + \frac{1}{\mu_0} B(-\nabla \times E)$$

We can then rewrite this:

$$\frac{\partial \mathcal{E}}{\partial t} = \frac{1}{\mu_0}\left[E \cdot (\nabla \times B) - B(\nabla \times E)\right] = \frac{1}{\mu_0} \nabla \cdot (B \times E)$$

We have seen that

$$\frac{d\mathcal{E}}{dt} = \frac{1}{\mu_0}\nabla \cdot (B \times E)$$

If we take the integral of this over a volume:

$$\int_V \frac{d\mathcal{E}}{dt}\, dV = \frac{1}{\mu_0}\int \nabla \cdot (B \times E)\, dV$$

This second integral is (from vector calculus) a surface integral:

$$\int \nabla \cdot (B \times E)\, dV = \oint B \times E\, dA$$

We can then rewrite the rate of change of energy:

$$\int \frac{d\mathcal{E}}{dt}\, dV = \frac{dw}{dt} = \frac{1}{\mu_0}\oint (B \times E)\, dA$$

where w is the energy. This means that the energy flow per unit area per unit time (the energy flux) is given by the Poynting vector:

$$S = \frac{1}{\mu_0}(E \times B)$$

Also note that often we refer to the magnetic field via H, where

$$H = \frac{B}{\mu}$$

And we can also use D to refer to the electric field:

$$D = \epsilon E$$

10.6 Proof of Momentum

Let us now look at the fact that EM waves impart momentum when they collide with things. This is a hand-wavy proof because we don't know the necessary math. Let's have an EM wave moving in the $\hat{z}$ direction, with $\vec{k} \propto (E \times B)$. We have some E field in the direction of $\hat{x}$, and a B field in the $\hat{y}$ direction. If we have an electric charge q sitting at the origin, we have certain forces on it:

$$F = q(E + v \times B)$$

The work done to the charge by the two fields will be

$$dW = F \cdot d\vec{r} = qE_x\, dx$$

We have no work done by the B field because $v \times B$ is normal to $d\vec{r}$. We can then use Newton's Law:

$$\frac{dp_z}{dt} = F$$

Note that we're only interested in the change in momentum in the same direction of the wave (in this case the $\hat{z}$ direction). If we had the particle moving in the $\hat{x}$ direction:

$$\frac{dp_z}{dt} = q(v_x \times B_y)$$

Which gives us that

$$\begin{aligned} dp_z &= qv_xB_y\,dt \\ &= qv_x\frac{E_x}{c}\,dt \\ &= q\frac{E_x}{c}\,v_xdt \\ &= q\frac{E_x}{c}\,dx \end{aligned}$$

Recalling that $dw = qE_x\,dx$, we see that

$$dp_z = \frac{dw}{c}$$

This tells us that the change in the momentum in the direction of propagation is the amount of work done divided by c. In relativity, there is the dispersion relation, which relates E and p:

$$E^2 = m^2c^4 + p^2c^2$$

where m is the mass of the particle. If we have that $p = \frac{E}{c}$, we have that $m = 0$. We have just discovered that EM waves (photons) have no mass, but can impart momentum.

10.7 Waves Changing Mediums

If we have a boundary between two mediums, what happens when a wave travels from one of the mediums to another. Let's first write Maxwell's equations for a medium (recalling that $D = \epsilon E$ and $H = \frac{B}{\mu}$):

$$\begin{gathered} \nabla\cdot E = \frac{\rho}{\epsilon_0} \to \nabla\cdot D = \rho \\ \nabla\cdot B = 0 \\ \nabla\times E = -\frac{dB}{dt} \\ \nabla\times B = \mu_0\epsilon_0\frac{dE}{dt} + \mu_0 J \to \nabla\times H = \frac{dD}{dt} + J \end{gathered}$$

We now have to think about the boundary conditions. We have done similar things with waves in strings, with parts of the wave transmitting through, and parts of the wave being reflected back. We also associated this with a property known as the impedance. We want to do something similar in this situation. We want to look at a narrow surface around the boundary. Maxwell's equations tell us that

$$\oint E\cdot dl = -\frac{d\Phi_B}{dt}$$

We know that the flux goes to 0 as $dx \to 0$. We have two bands for the surface, $E^1_{||}$ and $E^2_{||}$:

$$E^1_{||}l - E^2_{||}l = 0$$

This tells us that $E^1_{||} = E^2_{||}$.

We can apply the same argument for $\oint H\cdot dl = \epsilon\frac{d\Phi_E}{dt} + I$. What about the transverse component? We can once again create a surface, and we can use the equation

$$\int D\cdot A = \rho = 0$$

This gives us

$$D^1_\perp \Delta A - D^2_\perp \Delta A = 0$$

Giving us that

$$\epsilon_1 E^1_\perp = \epsilon_2 E^2_\perp$$

We can then use the same method for $\oint B \cdot A = 0$, which gives us that

$$B^1_\perp = B^2_\perp$$

To sum up, we have used Maxwell's equations at the boundary with no free current, we have that the parallel components of the electric field has to be continuous, the perpendicular component is proportional to the ratio between the ϵs. For the magnetic field, the perpendicular components are continuous, and the parallel components are proportional by a ratio of the μs. This sets us up to derive Snell's law, and we can then talk about why metals are shiny, why they're conductive, and talk about what happens when we put EM waves inside a hollow conductor (waveguides).

10.8 Waves in Conductors

We know that in a conducting medium, $\sigma \neq 0$, and in an insulator $\sigma = 0$. In a perfect conductor, $\sigma = \infty$. Let's talk about how EM waves interact with conductors, and why conductors have these certain properties. By Maxwell's Laws:

$$\nabla \times H = \frac{dD}{dt} + J$$

Writing this out:

$$\nabla \times H = \mathcal{E}\frac{dE}{dt} + \sigma E$$

Now taking a time derivative:

$$\frac{d(\nabla \times H)}{dt} = \mathcal{E}\frac{d^2E}{dt^2} + \sigma\frac{dE}{dt}$$

The left side then becomes

$$\nabla \times \frac{dH}{dt} = \frac{1}{\mu}\nabla \times \frac{dB}{dt}$$

By some vector calc identities:

$$= -\frac{1}{\mu}(\nabla(\nabla \cdot E) - \nabla^2 E)$$

This first term is 0 because we have no free charges:

$$= \frac{1}{\mu}\nabla^2 E$$

Putting this into the equation, we have that

$$\nabla^2 E = \mathcal{E}\mu\frac{d^2}{dt^2} + \mu\sigma\frac{dE}{dt}$$

Which gives us

$$\nabla^2 E = \frac{1}{c^2}\frac{d^2E}{dt^2} + \mu\sigma\frac{dE}{dt}$$

Thinking intuitively, we see that the second term leads to damping, and will lead to an imaginary portion in the $\vec{k}$ vector. We see that a conductor is something that applied damping, which makes sense.

Take for example a wave propagating in the $\hat{z}$ direction with $\vec{B}$ pointing in the $\hat{y}$ direction. Writing out the Laplacian in the wave equation, we see that the first two terms are zero, leaving us with

$$\frac{d^2E_x}{dx^2} = \mu\mathcal{E}\frac{d^2E_x}{dt^2} + \mu\sigma\frac{dE_x}{dt}$$

This has solution $E_x = E_0 e^{i\omega t}e^{-\gamma z}$, where

$$\frac{d^2E_x}{z^2} = (-\mu\mathcal{E}\omega^2 + \mu\sigma\omega)E_x$$

where the portion in parentheses is known as γ^2.

We have that

$$\frac{d^2E_x}{dz^2} - \gamma^2 E_x = 0$$

For a very good conductor:

$$\sigma > \mathcal{E}\omega \to \gamma^2 = \mu\omega\sigma i$$

Which gives us that

$$\gamma = (\mu\omega\sigma)^{1/2}\sqrt{i}$$

Where $\sqrt{i} = \frac{1+i}{\sqrt{2}}$. Placing this back into the solution:

$$E_x = E_0 e^{-(\frac{\mu\sigma\omega}{2})^{1/2}z} e^{i(\omega t - \frac{\mu\sigma\omega}{2})^{1/2}z)}$$

This gives us that

$$k = \left(\frac{\omega\mu\sigma}{2}\right)^{\frac{1}{2}}$$

This is called attenuation. Using this, we can find the phase velocity:

$$v_p = \frac{\omega}{k} = \left(\frac{2\omega}{\mu\sigma}\right)^{1/2}$$

The group velocity is the speed of propagation of energy:

$$v_g = \frac{d\omega}{dk} = \frac{1}{\frac{dk}{d\omega}}$$

We have previously said that the impedance in free space or in a medium is given by

$$Z = \sqrt{\frac{\mu}{\mathcal{E}}}$$

In the case of this plane wave, this is the same as

$$Z = \frac{E_x}{H_y}$$

What is the impedance when in a conducting medium?

$$E_x = E_0 e^{i\omega t} e^{-\gamma z}$$

$$H_y = H_0 e^{i(\omega t - \phi)} e^{-\gamma z}$$

$$\nabla \times E = -\frac{dB}{dt} = -\mu \frac{dH}{dt}$$

We have that

$$\frac{dE_x}{dz} - \frac{dE_z}{dx} = -\mu \frac{dH_y}{dt}$$

Keeping only the y terms:

$$\frac{dE_x}{dz} = -\mu \frac{dH_y}{dt}$$

Which gives us

$$Z = \frac{E_x}{H_y} = \frac{i\omega\mu}{\gamma}$$

This is a complex value, due to the i and γ:

$$Z = \frac{i\omega\mu}{(1+i)\left(\frac{\omega\mu\sigma}{2}\right)^{1/2}} = \left(\frac{\omega\mu}{\sigma}\right)^{1/2} e^{i\frac{\pi}{4}}$$

We can see that if $\sigma \to \infty$, $Z \to 0$. Another interesting thing is that a perfect conductor is also a perfect reflector, as we can show that as $\sigma \to \infty$, then $|\frac{I_t}{I_r}| \to 0$.

10.9 Waveguides

Let's now talk about wave guides, where waves are propagating through a hollow conductor, such as a metal pipe (in this case the pipe has a rectangular crosssection, and is a tall and b wide.). The wave's behavior is governed by Maxwell's equations and boundary conditions. We assume that the waveguide is a perfect conductor:

$$E = 0 \quad B = 0$$

When inside the conductor. We also know from boundary conditions that $E_{||} = 0$ and $B_\perp = 0$ right at the surface. We then have

$$\vec{E}(x, y, z) = \vec{E_0}(y, z) e^{i(kx - \omega t)}$$

And

$$\vec{B}(x, y, z) = \vec{B_0}(y, z) e^{i(kx - \omega t)}$$

Where

$$\vec{E_0} = E_x^0 \hat{x} + E_y^0 \hat{y} + E_z^0 \hat{z}$$

and similarly for $\vec{B_0}$. Now using Maxwell's equations:

$$\nabla \times E = -\frac{dB}{dt}$$

$$\left(\frac{dE_z}{dy} - \frac{dE_y}{dz}\right)\hat{x} + \left(\frac{dE_x}{dz} - \frac{dE_z}{dx}\right)\hat{y} + \left(\frac{dE_y}{dx} - \frac{dE_x}{dy}\right)\hat{z} = i\omega(B_x\hat{x} + B_y\hat{y} + B_z\hat{z})$$

We can now equate each term to the corresponding term on the other side. This then leaves us with

$$\frac{dE_z}{dy} - \frac{dE_y}{dz} = i\omega B_x \qquad \frac{dE_x}{dz} - ikE_z = i\omega B_y \qquad ikE_y - \frac{dE_x}{dy} = i\omega B_z$$

We can do the same for $\nabla \times B = \frac{1}{c^2}\frac{dE}{dt}$, and we follow the same procedure, that leads us to 3 equations:

$$\frac{dB_z}{dy} - \frac{dB_y}{dz} = -\frac{i\omega}{c^2}E_x \qquad \frac{dB_x}{dz} - ikB_z = -\frac{i\omega}{c^2}E_y \qquad ikB_y - \frac{dB_x}{dy} = -\frac{i\omega}{c^2}E_z$$

We can now take these 6 equations that we have, and solve for E_y,E_z, B_y, and B_z in terms of E_x and B_x:

$$E_y = \frac{i}{\left(\frac{\omega}{c}\right)^2 - k^2}\left[k\frac{dE_x}{dy} + \omega\frac{dB_x}{dz}\right]$$

$$E_z = \frac{i}{\left(\frac{\omega}{c}\right)^2 - k^2}\left[k\frac{dE_x}{dz} - \omega\frac{dB_x}{dy}\right]$$

$$B_y = \frac{i}{\left(\frac{\omega}{c}\right)^2 - k^2}\left[k\frac{dB_x}{dy} - \frac{\omega}{c^2}\frac{dE_x}{dz}\right]$$

$$B_z = \frac{i}{\left(\frac{\omega}{c}\right)^2 - k^2}\left[k\frac{dB_x}{dz} + \frac{\omega}{c^2}\frac{dE_x}{dy}\right]$$

If $E_x = 0$ and $B_x = 0$, we can immediately see that nothing travels, all the terms go to 0. This is called transverse EM (TEM), where there is no travelling wave in the waveguide. We can only have either transverse electric waves (TE waves), where $E_x = 0$ and $B_x \neq 0$; or transverse magnetic waves (TM waves), where $B_x = 0$ and $E_x \neq 0$.

We must also recall that E and B must satisfy:

$$\nabla \cdot E = 0 \qquad \nabla \cdot B = 0$$

This will give us two different wave equations:

$$\frac{d^2E_x}{dy^2} + \frac{d^2E_x}{dz^2} + \left[\left(\frac{\omega}{c}\right)^2 - k^2\right]E_x = 0$$

$$\frac{d^2B_x}{dy^2} + \frac{d^2B_x}{dz^2} + \left[\left(\frac{\omega}{c}\right)^2 - k^2\right]B_x = 0$$

So for a TE wave, we set $E_x = 0$, and we want to find B_x. This will completely determine the wave. We know that B_x is a function of y and z, and we have no easy way of solving for it. We can assume that it is made up of a product of two functions (this is separation of variables):

$$B_x(y, z) = Y(y)Z(z)$$

We can then plug these in:

$$\frac{d^2E_x}{dy^2} + \frac{d^2E_x}{dz^2} + \left[\left(\frac{\omega}{c}\right)^2 - k^2\right]E_x = 0$$

Turns into

$$Z\frac{d^2Y}{dy^2} + Y\frac{d^2Z}{dz^2} + \left[\left(\frac{\omega}{c}\right)^2 - k^2\right]YZ = 0$$

We can then divide by YZ:

$$\frac{1}{Y}\frac{d^2Y}{dy^2} + \frac{1}{Z}\frac{d^2Z}{dz^2} + \left[\left(\frac{\omega}{c}\right)^2 - k^2\right] = 0$$

We see that the first term only depends on y, the second term only depends on z, and the third term is a constant. Each of the terms is independent of the rest, so we can claim that the only way we can satisfy this for all values of z and y is that each of the terms are constant (you can think about this by setting one to a constant and seeing that it can only be satisfied if the other one also remains constant). This means that we can create two different equations now:

$$\frac{1}{Y}\frac{d^2Y}{dy^2} = -k_y^2 \qquad \frac{1}{Z}\frac{d^2Z}{dz^2} = -k_z^2$$

This leaves the original equation as

$$\boxed{-k_y^2 - k_z^2 + \left[\left(\frac{\omega}{c}\right)^2 - k^2\right] = 0}$$

These two subequations that we made are just like oscillator equations:

$$Y = A\sin(k_y y) + B\cos(k_y y)$$

$$Z = A'\sin(k_z z) + B'\cos(k_z z)$$

What about the boundary equations? We know that $B_\perp = 0$, which tells us that $B_y = 0$ (at $y = 0$ and $y = a$) and $B_z = 0$ (at $z = 0$ and $z = b$). We cna then plug these into the 4 equations we had, and we see that where $B_y = 0$, $\frac{dB_x}{dy} = 0$, and where $B_z = 0$, $\frac{dB_x}{dz} = 0$. This then tells us that

$$A\cos(k_y y) - B\sin(k_y y) = 0$$

Which then tells us that $A = 0$, which implies that $\frac{dB_x}{dy}$ (at $y = a$) is also 0, telling us that $k_Y a = n\pi$, telling us that $k_y = \frac{n\pi}{a}$. Replicating this for $\frac{dB_x}{dz} = 0$, giving us that $k_z = \frac{m\pi}{b}$.

This gets us that

$$B_x(y,z) = C\sin(\frac{n\pi}{a}y)\sin(\frac{m\pi}{b}z)e^{i(kx-\omega t)}$$

We also know that

$$k = \sqrt{\left(\frac{\omega}{c}\right)^2 - \pi^2\left(\frac{n^2}{a^2} + \frac{m^2}{b^2}\right)}$$

Note that if we want real valued k, we need $\omega > c\pi\sqrt{\frac{n^2}{a^2} + \frac{m^2}{b^2}}$. We see that the waveguide will only allow waves with this property, as lower frequencies will diminish (just plug in a complex k, and we see that the exponential term gets absorbed). The lowest acceptable frequency can be achieved if $a > b$, where $\omega = c\pi\frac{1}{a}$ ($n = 1$, $m = 0$). Looking at the phase velocity at an arbitrary frequency:

$$v_p = \frac{\omega}{k} = \frac{c}{\sqrt{1 - \frac{\omega_{nm}}{\omega}}}$$

Looking at the group velocity:

$$v_g = \frac{d\omega}{dk} = c\sqrt{1 - \left(\frac{\omega_{nm}}{\omega}\right)^2}$$

11 Wavefunctions

So far we have seen different types of wave equations. Wave equations give us the functions for the wave disturbance (y, E, B, etc.). What counts at the end is the energy flow, which is proportional to the magnitude of the disturbance squared. Let's now generalize the disturbance to any type of wave (could be y, E, etc.). We call the disturbance $\psi(x, y, z)$, and the intensity is $\propto |\psi|^2$.

Let's now try to get the intensity as a function of the distance from the source. Let's say we have a point source and a 3 dimensional wave propagating from the source spherically. What can we say about the intensity relative to the distance from the source? Well we know that the energy put out is conserved, so the total energy is given by

$$E_{total} = \int |\psi(r)|^2 \, dA$$

where r is the distance and A is the surface area of the sphere. We can see that ψ only depends on r:

$$E = |\psi(r)|^2 \int dA = 4\pi r^2 |\psi(r)|^2$$

We know that E is a constant, so

$$|\psi(r)|^2 \propto \frac{E_{total}}{4\pi r^2}$$

Which tells us that

$$\psi(r) \propto \frac{1}{r}$$

What about if we have a cylindrical source (a very long cylinder). We can create a cylinder around it (with radius r). We once again see that the total energy is a constant:

$$E_{total} = \int |\psi(r)|^2 \, dA$$

Since we have a very long cylinder, we can claim symmetry so ψ will only depend on r, and thus is a constant in the integral:

$$E_{total} = |\psi(r)|^2 \int dA = |\psi(r)|^2 (2\pi r l)$$

We see that

$$|\psi(r)|^2 = \frac{E_{total}}{2\pi r l} \to \psi(r) \propto \frac{1}{\sqrt{r}}$$

Note that if we block the source of a wave after the propagation begins, the wave will not be completely blocked because the wave has propagated to other locations, and those locations now affect other locations. We can think of this like generating a disturbance in a pond with an obstacle somewhere in the pond. You will see that there will still be waves generated/propagating through the areas that are beyond the obstacle.

This leads us into what is known as Huygen's Principle, which states that each point on a wave front acts as a source of new wave fronts (wavelets), and each wavefront is tangent to the surface of the new wavelets.

Fresnel added an additional statement that solved the issue of the wavelets going backwards into the source, where the wavelets have a cosine term:

$$\psi(r) = \frac{e^{-ikr}}{r}(\cos\theta + 1)$$

11.1 Double Slit Experiment

We want to take a distant source and two screens, the first of which has 2 slits in it and the second one with no slits (and is a distance D away from the first screen). The distance between the two slits is d. We assume that the slits are small relative to the wavelength of the emitted waves. Huygen's principle tells us that when a wave reaches one of the slits, it produces a new wave, $\psi_1 \propto \frac{1}{\sqrt{r_1}}$, and similarly for the other slit, $\psi_2 \propto \frac{1}{\sqrt{r_2}}$. Note that the slit is not a simple hole, but rather a cylindrical cutout (hence the $\sqrt{r}$). We will work in the far-field limit, which is when $D \gg d$. We have the total generated wave at some point P on the second screen:

$$\psi(r) = \psi(r_1) + \psi(r_2)$$

$$\psi(r) = \frac{a}{\sqrt{r_1}} e^{i(kr_1 - \omega t)} + \frac{a}{\sqrt{r_2}} e^{i(kr_2 - \omega t)}$$

We will say that $A_P = \frac{a}{\sqrt{r_1}} \approx \frac{a}{\sqrt{r_2}}$ (due to the far-field limit). Also note that the rays from the slits to the point P are both parallel due to the approximation that we are using, giving us that $r_2 - r_1 = d \sin\theta$. Even though we can ignore the difference in amplitude between r_2 and r_1, we can't ignore the difference in phase, because the difference $r_2 - r_1$ in the phase makes a pretty big difference. We are left with

$$\psi(P) = A_P e^{-\omega t} e^{ik(\frac{r_1+r_2}{2})} \left[e^{ik(\frac{r_1-r_2}{2})} + e^{-ik(\frac{r_1-r_2}{2})} \right]$$

Which becomes

$$\psi(P) = 2A_P \cos(k\frac{r_1 - r_2}{2}) e^{ik(\frac{r_1+r_2}{2} - \omega t)}$$

The amplitude of this:

$$A^P_{total} = 2A_P \cos(\frac{kd\sin\theta}{2}) = 2A_P \cos(\frac{\pi d \sin\theta}{\lambda})$$

We have a property of coherence, which is making sure that thelight coming out of slit 1 and slit 2 have constant phases throughout the propagation (they dont have to be the same, but they should produce constant phases). This is why we put in only 1 source, so the light reaching the two slits is from the same source.

We have that

$$A_{tot}(\theta) = A(\theta) \cos\left(\frac{kd\sin\theta}{2}\right)$$

Normally we are interested in the intensity relative to $\theta = 0$, i.e how the wave changes from the center of the slit as we move towards where the light strikes the second screen:

$$\frac{A_{tot}(\theta)}{A_{tot}(0)} = \frac{A(\theta)}{A(0)} \cos\left(\frac{kd\sin\theta}{2}\right)$$

Comparing the intensities:

$$\frac{I(\theta)}{I(0)} = \left|\frac{A(\theta)}{A(0)}\right|^2 \cos^2\left(\frac{kd\sin\theta}{2}\right)$$

Essentially,

$$\frac{I(\theta)}{I(0)} \propto \cos\theta \cos^2\left(\frac{kd\sin\theta}{2}\right)$$

Using the diagram of the experiment, we can see that this turns into

$$\frac{I(\theta)}{I(0)} = \frac{D}{\sqrt{x^2 + D^2}} \cos^2 \left[\frac{\pi}{\lambda}\left(\frac{dx}{\sqrt{x^2 + D^2}}\right)\right]$$

This causes the creation of fringes as we approach the screen. If we do some more approximations, we see that for small θ or small $\frac{x}{D}$, it turns into

$$\frac{I(\theta)}{I(0)} = \cos^2\left(\frac{\pi}{\lambda} d\theta\right) = \cos^2\left(\frac{\pi}{\lambda}\frac{dx}{D}\right)$$

11.2 N-slit Interference

If we have N slits separated by a distance of d, and the second screen a distance D away. The overall result of the waves from all of the slits becomes:

$$\psi(\theta) = \sum_{n=1}^{N} \psi_n(\theta) = \sum_{n=1}^{N} A(\theta) e^{i(kr_n - \omega t)}$$

We now make the approximation that $r_1 \approx r_2 \approx r_3 \cdots \approx r_n$ (but only in the amplitude, not in the phase):

$$\psi(\theta) = A(\theta) e^{i(kr_1 - \omega t)} \sum_{n=1}^{N} e^{ik(r_n - r_1)}$$

We call the quantity $\sum_{n=1}^{n} e^{ik(r_n - r_1)}$ ϕ. We have that

$$\phi = \sum_{n=1}^{N} e^{ik(r_n - r_1)}$$

We know that $r_n - r_1 = (n-1)d\sin\theta$:

$$\phi = \sum_{n=1}^{N} e^{ik(n-1)d\sin\theta}$$

If we have a quantity $z = e^{ikd\sin\theta}$:

$$\phi = \sum_{n=0}^{N-1} z^n = \frac{z^N - 1}{z - 1} = \frac{e^{iNdk\sin\theta} - 1}{e^{ikd\sin\theta} - 1}$$

Factoring this:

$$\phi = e^{ik(N-1)\sin\theta} \frac{\sin(\frac{Nkd\sin\theta}{2})}{\sin(\frac{kd\sin\theta}{2})}$$

Plugging this back into the definition of ψ:

$$\psi = A(\theta) \frac{\sin(\frac{Nkd\sin\theta}{2})}{\sin(\frac{kd\sin\theta}{2})} \left[e^{\frac{ik(N-1)d\sin\theta}{2}} e^{i(kr_1 - \omega t)}\right]$$

Looking at the relative intensity (where we've set $\alpha = kd\sin\theta$):

$$\frac{I(\theta)}{I(0)} = \left|\frac{A(\theta)}{A(0)}\right|^2 \left|\frac{\phi(\theta)}{\phi(0)}\right|^2$$
$$= \frac{1}{N^2}|\frac{A(\theta)}{A(0)}|^2 \left|\frac{\sin\frac{N\alpha}{2}}{\sin\frac{\alpha}{2}}\right|^2$$

We have the function

$$\frac{1}{N^2}|\frac{A(\theta)}{A(0)}|^2 \left|\frac{\sin\frac{N\alpha}{2}}{\sin\frac{\alpha}{2}}\right|^2$$

We see that this is a periodic function, and we know that

$$\frac{N\alpha}{2} = m\pi$$

From this, we see that if both the numerator and the denominator of the second term approach 0, we are left with

$$\frac{I(\theta)}{I(0)} = \left|\frac{A(\theta)}{A(0)}\right|^2$$

Which gives us maxima. Looking at a plot of this function, we see that where θ is a multiple of 2π, we have constructive interference of all the slits. If just the numerator is 0, we have that

$$\alpha = \frac{2\pi m}{N}, \quad = 1, 2, \ldots, N-1$$

This provides the locations of the minima. Note that the number of minima in between each maximum is given by $N-1$, and the larger the N the lower the size of the minima. When we take the case where $N \to \infty$, the local minima vanish, and the maxima become extremely sharp. This leads into the resolving power of our interferometer. We can derive the resolving power of an N-slit interferometer:

$$d\sin\theta = m\lambda$$

Making both sides differentials:

$$\Delta(d\sin\theta) = m\Delta\lambda$$

And now rewriting:

$$\Delta\theta = \frac{m}{d}\Delta\lambda$$

This is one relationship that we have. We also have the relationship

$$\Delta\alpha = \Delta(\frac{2\pi}{\lambda}\sin\theta) = \frac{2\pi}{N}$$

This then leaves us with

$$\Delta\theta = \frac{\lambda}{Nd\cos\theta}$$

Setting the two expressions for $\Delta\theta$ together, we have that

$$\frac{m\Delta\lambda}{d\cos\theta} = \frac{\lambda}{Nd\cos\theta}$$

We then find that

$$\frac{\lambda}{\Delta\lambda} = mN$$

This is the resolving power of the N-slit interferometer. For best separation, we need large N and at high orders.

11.3 Diffraction

Diffraction occurs when we have a wider slit, and we don't just have a single wave coming in, we have many. We'll now need to integrate all wavelets to get the sum at some point P on the second screen. An easy solution is to use what we just had and take $N \to \infty$ and let the slit separation increase, turning it into one big wide slit. If we plug in the situation, we see that we are left with

$$\frac{I(\theta)}{I(0)} = \left| \frac{\sin \frac{\beta}{2}}{\frac{\beta}{2}} \right|^2$$

where $\beta = ka \sin\theta$. This then leaves us with:

$$\sin\theta = \frac{\lambda}{a}$$

This is the half-width of the diffraction profile.

We can also use the Huygen's principle and do a formal derivation. Any wave coming from the slit is given by

$$\psi(r) = \frac{1}{r} e^{i(kr - \omega t)}$$

The wave at some point P on the second screen (once again saying that it is far enough that the lines are parallel):

$$\psi_P = \int \left(\frac{1}{r} e^{i(kr-\omega t)} \right) C(y)\, dA$$

where $C(y)$ is the intensity. Rewriting this (using the fact that $kr = k(r_0 - y \sin t)$, and in the case where $C(y)$ is a constant c, and finally that $\frac{1}{r} \approx \frac{1}{r_0}$):

$$\psi_P(r) = \frac{CL}{r_0} e^{i(kr_0 - \omega t)} \int_{-\frac{a}{2}}^{\frac{a}{2}} e^{-iky\sin\theta}\, dy$$

Where L is the length of the slit. Compressing the constants into one factor G:

$$= G \frac{1}{ik\sin\theta} \left(e^{-\frac{ika\sin\theta}{2}} - e^{\frac{ika\sin\theta}{2}} \right)$$

This then leaves us with

$$-2G\frac{a}{2} \left[\frac{\sin \frac{ka\sin\theta}{2}}{\frac{ka\sin\theta}{2}} \right]$$

Recall that we called $\beta = ka\sin\theta$, and $I \propto |\psi|^2 \propto \left| \frac{\sin\frac{\beta}{2}}{\frac{\beta}{2}} \right|^2$. We see that we get the same thing we did before.

Now lets talk about the effect of diffraction on interference patterns. The diffraction actually "smears" the interference patterns, and as the slit gets smaller $a \to 0$, the effect of the smear actually increases. Instead, we need to increase a to decrease the diffraction effects, but that starts interfering with some of our assumptions. What we can do is increase λ as well, and that will help decrease the effect of the diffraction on the interference pattern.

Let's say we have a rectangular slit. If we take a arbitrary point on the surface of the aperture r':

$$\psi_P(r) = G \iint e^{-i(k\hat{r}\cdot r')}\, dA$$

We are using the technique that we did for a single wide slit, and expanding it via a double integral over the area. If we do this integral out, we see that we are left with

$$= G\left(\frac{\sin\frac{\alpha}{2}}{\frac{\alpha}{2}}\right)^2\left(\frac{\sin\frac{\beta}{2}}{\frac{\beta}{2}}\right)^2$$